SAMPLE FUNCTIONS OF GAUSSIAN RANDOM HOMOGENEOUS FIELDS ARE EITHER CONTINUOUS OR VERY IRREGULAR

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1. In [1] Yu. K. Belyaev proves the following is true of any stationary stochastic process $t \to x(t)$ over the real numbers, mean-square continuous as a function of the reals, and having Gaussian joint distributions: Either with probability one the paths of the process are continuous or with probability one thsey are totally unbounded in any time interval. In this paper we will establish in a impler way the analogous result for a left-homogeneous random field over a second-countable locally compact topological group; i.e., one whose topology contains a countable base and is locally compact. We may arrange and re-label Belyaev's results as follows:

Lemma 1.1. *If the map $t \to x(t)$ fails at t_0 to be continuous with probability one, then there exists $a > 0$ such that for all t_0 and every neighborhood N of t_0,*

$$\sup_{t \in N} (x(t) - x(t_0)) > a \quad \textit{with probability one.} \tag{1}$$

Lemma 1.2. *If $a > 0$ exists such that (1) holds for all t_0 and all neighborhoods N of t_0, then for all t_0 and N $\sup_{t \in N} (x(t) - x(t_0)) > 2a$ with probability one.*

By induction, if condition (1) holds then given any positive M, for every N, $\sup_{t \in N} (x(t) - x(t_0)) > M$ with probability one. Belyaev's proofs of both lemmas depend on the fact that one can write $x(t) = \sum_{n \geq 0} y_n(t)$, where for $i \neq j$, $y_i(s)$ and $y_j(t)$ are independent Gaussian random variables no matter what s and t, and where each summand $t \to y_n(t)$ is continuous with probability one.

Belyaev's proof of Lemma 1.1 is difficult and depends on the fact that each summand $t \to y_n(t)$ can be made stationary. This he does by regarding each $x(t)$ as a stochastic integral over the dual group of the reals, and by partioning the domain of integration into countably many compact subsets. Although the random variables of random homogeneous fields over many interesting groups can also be so regarded (i.e., as random integrals over the dual object of the group), we wish to avoid irrelevant and difficult questions such as: When is the support of a spectral measure small enough to ensure path continuity with probability one, and when can one partition the dual object into countably many such "small" subsets?

On the other hand, Belyaev's proof of Lemma 1.2 applies to arbitrary locally compact second-countable groups with only minor obvious changes of notation. The proof is given here as Lemmas 2.4 and 2.5. It is essentially a verbatim repeat of the corresponding part of Belyaev's proof.

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