A THEOREM OF LÉVY AND A PECULIAR SEMIGROUP[1]

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1. Introduction. Let I be a finite or countably infinite set. For each $t \geqq 0$ let $P(t)$ be a stochastic matrix on I, such that $P(t+s) = P(t)P(s)$, $P(0)$ is the identity matrix, and $P(t) \to P(0)$ coordinatewise as $t \to 0$. Then P is called a standard stochastic semigroup on I.

The result of Lévy (1958) referred to in the title is:

(1) Theorem. *For each pair i, j with $i \neq j$, there are only two possibilities: either $P(t, i, j) = 0$ for all $t \geqq 0$, or $P(t, i, j) > 0$ for all $t > 0$.*

One object of this note is to sketch an alternative proof of this fact. For historical discussion and some of the known proofs, see (Chung, 1960).

As is well known, P has a coordinatewise derivative at 0, called the infinitesimal generator Q. Another object of this note is to sketch the construction which proves

(2) Theorem. *There is a standard stochastic semigroup P on $I = \{1, 2, \cdots\}$ whose infinitesimal generator Q is given by:*

$$Q(i, i) = -\infty \quad \text{for all} \quad i \quad \text{in} \quad I \tag{3}$$

$$Q(i, j) = 0 \quad \text{for all} \quad i \neq j \quad \text{in} \quad I. \tag{4}$$

These results are discussed together because they involve the same technique, restricting a Markov chain to a subset of its state space. For simplicity, suppose all states are recurrent.

2. Restricting a Markov chain. Give I the discrete topology, and let $\bar{I} = I$ when I is finite, $\bar{I}$ = one point compactification of I when I is infinite. Let $\{X(t): 0 \leqq t < \infty\}$ be an $\bar{I}$-valued stochastic process on a probability triple $(\Omega, \mathfrak{F}, \mu)$, which is a Markov chain with stationary standard transitions P. For technical safety, suppose the sample functions of X are quasiregular (the definition is in Section 5). Let J be a finite subset of I, and let X_J be the *restriction* of X to J, that is, X watched only when in J. More formally, let $\tau_J(t)$ be the greatest s such that the Lebesgue measure of $\{u: 0 \leqq u \leqq s, X(u) \in J\}$ is t. Then $X_J(t) = X[\tau_J(t)]$. From the strong Markov property, X_J is a Markov chain with stationary transitions, call them P_J. Plainly, P_J is a standard stochastic semigroup on J. Call its infinitesimal generator Q_J, and say that P_J (respectively, Q_J) is P (respectively, Q) restricted to J. Plainly, for $K \subset J$, $(P_J)_K = P_K$ and $(Q_J)_K = Q_K$. It is not hard to check that

$$Q \leqq Q_J. \tag{5}$$

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