ON A STOPPING RULE AND THE CENTRAL LIMIT THEOREM[1]

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Let $S_n = \sum_{k=1}^n x_k$, $n \geqq 1$, where x_k, $k \geqq 1$, are independent, orthonormal (i.o.) random variables, i.e., $Ex_k = 0$, $Ex_k^2 = 1$, $k \geqq 1$. For $c > 0$, let t_c be the smallest positive integer n such that $|S_n| > cn^{\frac{1}{2}}(= \infty$ if no such n exists). Our principal aim is to prove the following:

Theorem 1. *If*

$$\lim_{n\to\infty} P(S_n/n^{\frac{1}{2}} \leqq x) = \Phi(x), \tag{1}$$

where $\Phi(x)$ *is the standard normal distribution, then*

$$E(t_c) < \infty \quad \text{if} \quad 0 \leqq c < 1; \tag{2}$$

$$E(t_c) = \infty \quad \text{if} \quad c \geqq 1. \tag{3}$$

Special cases of this theorem have appeared previously. Blackwell and Freedman [1] have treated the "coin-tossing" case in which x_k, $k \geqq 1$, are symmetric and assume the values ± 1. Chow, Robbins, and Teicher [3] have proved that (2) holds for i.o. sequences that are uniformly bounded and that (3) holds for *arbitrary* i.o. sequences. When the variables are identically distributed and $E|x_1|^3 < \infty$, one may conclude that (2) holds, at least for c sufficiently small, by using estimates of tail probabilities, due to Breiman [2].

Our computations rely on the well-known Lindeberg-Feller theorem:

Condition (1) *is equivalent to*

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^n \int_{\{x_k^2 > \epsilon n\}} x_k^2 \, dP = 0 \tag{4}$$

for every $\epsilon > 0$.

Since Chow *et al.* have shown that (3) holds for every sequence of i.o. variables, it suffices to show that $E(t_c) < \infty$ for $0 < c < 1$. We now fix c and so may dispense with it as a subscript in the sequel. In what follows, we deal with the sequence of stopping rules $\tau = \tau(n) = \min(t, n)$, $n \geqq 1$. Then by a variant of Wald's lemma $ES_\tau^2 = E\tau$, $n \geqq 1$, so that

$$\text{Et} = \lim_{n\to\infty} E\tau = \lim_{n\to\infty} ES_\tau^2 \leqq \infty.$$

(See Section 2 of [3].) The following lemma says, in effect, that instead of considering the expectation of the square of the entire random sum, it is sufficient for our purposes to consider only Ex_τ^2, the expectation of the square of the final term.

Received 15 June 1967.

[1] The second author acknowledges partial support for this research by the National Science Foundation, NSF GP-3694 at Columbia University.

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