A FIRST PASSAGE PROBLEM FOR THE WIENER PROCESS

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We study the stopping time $T = T_{a,c}$, the first time the Wiener process $W(t)$, $0 \leqq t < \infty$ crosses the curve $\pm c(t + a)^{\frac{1}{2}}$, $a > 0$, $c > 0$. In an analogous discrete time problem the moments of T were studied in Blackwell-Freedman (1964), Chow-Robbins-Teicher (1965), and Chow-Teicher (1966). Results obtained showed that for the discrete problem $ET < \infty$ if and only if $c < 1$, and $ET^2 < \infty$ if and only if $c < (3 - 6^{\frac{1}{2}})^{\frac{1}{2}}$. We show that these statements are also valid in our case and find the generalization: $ET^n < \infty$ if and only if c is less than the first positive zero of the Hermite polynomial He_{2n}. We conjecture that the same result holds also in the original discrete case. (Note that $(3 - 6^{\frac{1}{2}})^{\frac{1}{2}}$ is the first zero of He_4.) We also give explicit formulas for the moments of T. Our method is based on the Wald identity for the Wiener process: $E \exp(-\lambda^2 T/2 + \lambda W(T)) = 1$.

We show that *any* stopping time T with $ET < \infty$ satisfies $EW(T) = 0$ and $EW^2(T) = ET$. These identities will be derived as a consequence of the basic properties of the Ito stochastic integral.

1. Stopping times and Ito integrals. Let $W(t, \omega)$, $0 \leqq t < \infty$, $\omega \varepsilon \Omega$ be a Wiener process with continuous paths. A finite nonnegative rv $T(\omega)$ is called a stopping time if $\{T \leqq t\} \varepsilon \mathfrak{S}\{W(s): s \leqq t\}$, $0 \leqq t < \infty$. For such a T let $\varphi_T(t, \omega) = 1$ for $t \leqq T(\omega)$, $\varphi_T(t, \omega) = 0$ for $t > T(\omega)$.

Lemma 1. *The Ito integral* $I(\varphi_T)(\omega) = \int_0^\infty \varphi_T(t, \omega)\, dW(t)$ *is defined and* $I(\varphi_T) = W(T)$ a.s.

Proof. According to K. Ito (1951), we must show that φ_T is measurable in $t \times \omega$, is nonanticipative, and satisfies $\int_0^\infty \varphi^2(t, \omega) dt < \infty$ a.s. Let $T_n = k/2^n$ on $\{(k - 1)/2^n \leqq T < k/2^n\}$. It is easy to check that T_n is both measurable and nonanticipative and that $\varphi_{T_n} \to \varphi_T$ at each (t, ω). Since $T(\omega) = \int_0^\infty \varphi^2(t, \omega)\, dt$ and $T(\omega) < \infty$ a.s. because T is a stopping time, we have checked that $I(\varphi_T)$ is defined. That $I(\varphi_T) = W(T)$ is a consequence of the definition of the Ito integral and the continuity of Wiener paths.

Theorem 1. *Let T be any stopping time with $ET < \infty$. Then $EW(T) = 0$ and $EW^2(T) = ET$.*

Proof. The mean of an Ito integral with finite variance is zero (Ito (1951)). The variance of $I(\varphi_T) = W(T)$ is given by Ito's formula $\int_0^\infty E\varphi_T^2(t, \omega)\, dt = \int_0^\infty P\{T \geqq t\}\, dt = ET$. This proves both assertions.

Caution. It is possible that $ET = \infty$ and $EW^2(T) < \infty$ as the example $T =$ time of crossing level one shows.

The result for discrete time corresponding to Theorem 1 has been proved in great generality by martingale methods [3].

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