ASYMPTOTICALLY OPTIMAL TESTS FOR MULTIVARIATE NORMAL DISTRIBUTIONS

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1. Introduction. Early work on probabilities of large deviations had to do with probabilities of deviations of the sample mean as the sample size increased. The papers of Chernoff [3] and Bahadur and Rao [1] are notable in this area. More general sets were considered by Borokov and Rogozin [2], Hoeffding [4], and Sanov [7]. Sanov [7] proved a result concerning probabilities of sets contained in the space of maximum likelihood estimates of the parameters of a multinomial distribution as the sample size increased. His result was of consequence only when the true parameter point was sufficiently "far" from the set in question. Hoeffding [5] sharpened this result to the following. If Ω is defined by

$$\Omega = \{(x_1 \cdots, x_k) : x_1 \geqq 0, \cdots, x_k \geqq 0; x_1 + \cdots + x_k = 1\}$$

and $z^{(N)}$ as the k-vector of maximum likelihood estimates of the k-vector $\mathbf{p}$ of parameters for a k dimensional multinomial distribution based on a sample size N; then the probability $P_N(A \mid \mathbf{p})$ that $z^{(N)} \varepsilon A \subset \Omega$ is given by

$$P_N(A \mid \mathbf{p}) = \exp\left[-NI(A^{(N)}, \mathbf{p}) + O(\log N)\right]$$

where

$$I(A^{(N)}, \mathbf{p}) = \inf\{I(\mathbf{x}, \mathbf{p}) : \mathbf{x}\, \varepsilon\, A^{(N)}\}$$

for

$$I(\mathbf{x}, \mathbf{p}) = \textstyle\sum_{i=1}^{k} x_i \log(x_i/p_i),$$

if $\mathbf{p} = (p_1, \cdots, p_k)$, $\mathbf{x} = (x_1, \cdots, x_k)$; uniformly in A and $\mathbf{p}$. Here $A^{(N)}$ is the intersection of the space of maximum likelihood estimates, $z^{(N)}$, with A.

Applying this estimate to the error probabilities involved in testing simple and composite hypotheses, Hoeffding was able to substantiate the following proposition: "If a given test of size α_N is 'sufficiently different' from a likelihood ratio test, then there is a likelihood ratio test of size $\leqq \alpha_N$ which is considerably more powerful than the given test at 'most' points $\mathbf{p}$ in the set of alternatives when N is large enough, provided that $\alpha_N \to 0$ at a suitable rate."

This result depends almost entirely on the probability estimate. It is conjectured by Dr. Hoeffding that such an estimate holds for a wide class of distributions of exponential type. The present work is an attempt to extend the probability result to the class of non-singular multivariate normal distributions and to apply this result in a way analogous to that of Hoeffding [5] to the problem of comparing tests of simple and composite hypotheses with appropriate likeli-

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