AN INEQUALITY FOR EXPECTED VALUES OF SAMPLE QUANTILES[1]

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1. Introduction. Let F be a continuous distribution function on R^1 that is strictly increasing on the (finite or infinite) open interval I where $0 < F < 1$, and let G denote the inverse of F. For $n = 1, 2, \cdots$ and $0 < \lambda < 1$, let

$$(1.1) \quad \gamma_n(\lambda) = [\Gamma(n+1)/\Gamma(\lambda(n+1))\Gamma((1-\lambda)(n+1))] \int_0^1 G(y) y^{\lambda(n+1)-1} \cdot (1-y)^{(1-\lambda)(n+1)-1}\, dy.$$

Obviously, if $X_{i:n}$ denotes the ith order statistic of a sample of size n from the parent distribution F, then

$$\gamma_n(i/(n+1)) = EX_{i:n}, \qquad i = 1, 2, \cdots, n.$$

We shall call $\gamma_n(\lambda)$ the expected value of the λ-quantile of a sample of size n from F, even though this interpretation is meaningless when $\lambda(n+1)$ is not an integer. We shall assume that for some λ the integral converges for sufficiently large n, which ensures that the same will hold for every $0 < \lambda < 1$. By making minor changes in W. Hoeffding's proof in [2], one shows that γ_n converges to G on $(0, 1)$ for $n \to \infty$.

Consider another continuous distribution function F^* that is strictly increasing on the interval I^* where $0 < F^* < 1$, and let G^*, γ_n^* and $X_{i:n}^*$ be defined for F^* analogous to G, γ_n and $X_{i:n}$ for F. Furthermore let

$$(1.2) \quad \phi(x) = G^*F(x), \qquad x \varepsilon I.$$

In [5] the author studied the following order relations between F and F^*:

(1.3) ϕ is convex on I;

(1.4) F and F^* represent symmetric distributions and ϕ is concave-convex on I.

If x_0 denotes the median of F, relation (1.4) implies that ϕ is antisymmetric about x_0 (i.e. $\phi(x_0 + x) + \phi(x_0 - x) = 2\phi(x_0)$) and hence that ϕ is concave for $x < x_0$ and convex for $x > x_0$.

Let ϕ_n be the function that maps the expected value of the λ-quantiles of a sample of size n from F on the corresponding quantities for F^*:

$$(1.5) \quad \phi_n(x) = \gamma_n^* \gamma_n^{-1}(x).$$

Received 21 November 1966; revised 12 June 1967.

[1] Report S 369, Mathematisch Centrum, Amsterdam.