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A NOTE ON THE ABSENCE OF TANGENCIES IN GAUSSIAN SAMPLE PATHS[1]

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Let $Y(t)$, $t \varepsilon [0, 1]$, be a Gaussian process having continuous sample paths with probability one, and let the mean and standard deviation of $Y(t)$ be denoted by $m(t)$ and $\sigma(t)$, respectively. In this Gaussian setting, quadratic mean continuity follows continuity with probability one and hence the functions m and σ, as specified above, are continuous on [0, 1]. It is assumed that $\min_{0 \leqq t \leqq 1} \sigma(t) > 0$.

Suppose u is a fixed continuous function on [0, 1]. We will say Y is somewhere tangent to u if there is an open interval $I \subset [0, 1]$ on which $Y - u$ has a constant sign and a $t \varepsilon I$ for which $Y(t) - u(t) = 0$ (intervals of the form $[0, a)$ and $(b, 1]$ are taken as open). It will be convenient to think of tangencies as being from above or from below, from above being associated with $Y - u$ non-negative on I. The purpose of this note is to show that Y is somewhere tangent to u with probability zero. This is known for stationary Gaussian processes with $m \equiv 0$, $u \equiv 0$ and $\sigma \not\equiv 0$, [2]. A simple modification of the proof in [2] shows it true for $u - m$ and σ constant functions, $\sigma \not\equiv 0$. The result is also known when, essentially, Y has a quadratic mean derivative at each $t \varepsilon [0, 1]$, $\min_{0 \leqq t \leqq 1} \sigma(t) > 0$, and u is continuously differentiable [1], pg. 289. After a reduction of the problem, much reliance is placed on the method of proof used in [2].

Consider a new process defined by

$$X(t) = [Y(t) - u(t)]/\sigma(t) + \lambda, \qquad t \varepsilon [0, 1],$$

with λ a constant to be fixed. $X(t)$, $t \varepsilon [0, 1]$, is a Gaussian process having continuous sample paths with probability one and

$$\bar{m}(t) = EX(t) = [m(t) - u(t)]/\sigma(t) + \lambda, \qquad \bar{\sigma}(t) = [E(X(t) - \bar{m}(t))^2]^{\frac{1}{2}} \equiv 1.$$

Let λ be chosen so large that $\bar{m}$ is non-negative on [0, 1]. Evidently tangencies of Y to u from above (below) are reflected in tangencies of X to the constant function λ from above (below).

We show X is somewhere tangent to λ from below with probability zero. This follows by first noting that some I in the definition of a tangency may be replaced by some I from among a fixed countable collection of intervals (see the remark above Lemma 1 of [2]). Second, for any given I, X is tangent to λ from below on I with probability zero. For, such an event implies that $\sup_I X(\cdot) = \lambda$ and we have the

Lemma. $\sup_I X(\cdot)$ *has a continuous distribution for any open interval* $I \subset [0, 1]$.

Proof. It is first claimed that if $T = \{t_1, \cdots, t_n\}$ is a finite set drawn from

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