

NOTES

ESTIMATING THE TOTAL PROBABILITY OF THE UNOBSERVED OUTCOMES OF AN EXPERIMENT¹

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An experiment has the possible outcomes E_1, E_2, \dots with unknown probabilities p_1, p_2, \dots ; $p_i \geq 0, \sum_i p_i = 1$. In n independent trials suppose that E_i occurs x_i times, $i = 1, 2, \dots$, with $\sum_i x_i = n$. Let $\varphi_i = 1$ or 0 according as $x_i = 0$ or $x_i \neq 0$. Then the random variable $u = \sum_i p_i \varphi_i$ is the sum of the probabilities of the unobserved outcomes. How can we "estimate" u ? (The quotation marks appear because u is not a parameter in the usual statistical sense.)

Suppose we make one more independent trial of the same experiment, and that in the total of $n + 1$ trials E_i occurs y_i times, $i = 1, 2, \dots$, with $\sum_i y_i = n + 1$. (Each $y_i = x_i$, except for one value of the subscript.) Let $\psi_i = 1$ or 0 according as $y_i = 1$ or $y_i \neq 1$. Consider the statistic $v = (n + 1)^{-1} \sum_i \psi_i$, which is the number of "singleton" outcomes of the $n + 1$ trials, divided by $n + 1$. In contrast to u , v is observable. The idea of using something like v to estimate u goes back to A. M. Turing according to [1], where the problem is discussed from a somewhat different point of view. We shall show that v is a good "estimator" of u in the sense that setting $w = u - v$ we have always

$$Ew = 0, \quad Ew^2 < (n + 1)^{-1}.$$

In fact,

$$Ew = \sum_i (p_i q_i^n - (n + 1)(n + 1)^{-1} p_i q_i^n) = \sum_i 0 = 0,$$

and a little algebra shows that $(n + 1) Ew^2$ is equal to the expression

$$(1) \quad \sum_i p_i q_i^n (1 + (n - 1)p_i) - \sum_{i \neq j} p_i p_j (1 - p_i - p_j)^n.$$

Hence since $1 - x \leq e^{-x}$,

$$(n + 1)Ew^2 \leq \sum_i p_i e^{-n p_i} e^{(n-1)p_i} = \sum_i p_i e^{-p_i} < \sum_i p_i = 1.$$

In the special case in which some k of the p_i are equal to $1/k$ and all the others are 0, the expression (1) reduces to

$$(2) \quad (1 + (n - 1)k^{-1})(1 - k^{-1})^n - (1 - k^{-1})(1 - 2k^{-1})^n.$$

Set

$$a_n = \sup (1) \text{ for all probability vectors } (p_1, p_2, \dots),$$

$$b_n = \sup (2) \text{ for all } k = 1, 2, \dots;$$

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