The Annals of Mathematical Statistics
1968, Vol. 39, No. 1, 123–133

CONVERGENCE OF SUMS OF SQUARES OF MARTINGALE DIFFERENCES[1]

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1. Introduction and notation. Let $(\Omega, \mathfrak{F}, P)$ be a probability space. A stochastic basis $(\mathfrak{F}_n, n \geqq 1)$ is a monotonically increasing sequence of σ-fields of measurable sets. A stochastic sequence $(y_n, \mathfrak{F}_n, n \geqq 1)$ consists of a stochastic basis $(\mathfrak{F}_n, n \geqq 1)$ and a sequence of random variables $(y_n, n \geqq 1)$ such that y_n is $\mathfrak{F}_n$-measurable. For a stochastic sequence $(x_n, \mathfrak{F}_n, n \geqq 1)$, we put (here as well as in following sections)

$$x_0 = 0, \mathfrak{F}_0 = \{\Phi, \Omega\}, d_n = x_n - x_{n-1} \quad \text{for} \quad n \geqq 1, s_n = (\textstyle\sum_{k=1}^n d_k^2)^{\frac{1}{2}},$$

$$x^* = \sup_{n \geqq 1} |x_n|, d^* = \sup_{n \geqq 1} |d_n|, s = \lim_{n \to \infty} s_n,$$

and I_A = indicator function of set A. If $(x_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale, then $(d_n, \mathfrak{F}_n, n \geqq 1)$ is called a martingale difference sequence. For a given stochastic basis $(\mathfrak{F}_n, n \geqq 1)$, a stopping time t is an extended positive integral valued measurable function such that $[t = n] \varepsilon \mathfrak{F}_n$ for each n. For a stopping time t and a measurable function y, $E_t y$ is defined as $\int_{[t<\infty]} y \, dP$ (or $\int_{[t<\infty]} y$, in short), if it exists.

Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale. Austin [1] recently proves that if $\sup_{n \geqq 1} E|x_n| < \infty$, $s < \infty$ a.e.; also Burkholder [2] proves that if $Es < \infty$ x_n converges a.e. and that if $\sup_{n \geqq 1} E|x_n| < \infty$, then $\sum_{k=1}^\infty \varphi_k d_k$ converges a.e. for every stochastic sequence $(\varphi_k, \mathfrak{F}_{k-1}, n \geqq 1)$ for which $\sup_{n \geqq 1} |\varphi_n| < \infty$ a.e.; Gundy [8] proves that if $(d_n, n \geqq 1)$ is an orthonormal sequence such that each d_n assumes at most two non-zero values with positive probability, and if the σ-field generated by $d_1, \cdots, d_n$ consists of exactly n atoms, such that

$$\inf_{n \geqq 1} \min(P[d_n > 0], \quad P[d_n < 0])/P[d_n \neq 0] > 0,$$

then for every sequence a_n of real numbers, $\sum_{n=1}^\infty a_n^2 d_n^2 < \infty$ if and only if $\sum_{n=1}^\infty a_n d_n$ converges.

Let $(\mathfrak{F}_n, n \geqq 1)$ be a stochastic basis. If for each n, $\mathfrak{F}_n$ is generated by atoms of $\mathfrak{F}_n$, then $(\mathfrak{F}_n, n \geqq 1)$ is said to be atomic. For a σ-field $\mathcal{G}$ of measurable sets and $A \varepsilon \mathfrak{F}$, a $\mathcal{G}$-measurable cover of A is a set $C \varepsilon \mathcal{G}$ such that $P(A - C) = 0$ and that if $B \varepsilon \mathcal{G}$ and $P(A - B) = 0$, then $P(C - B) = 0$. For $A \varepsilon \mathfrak{F}$, let $C_n(A)$ be the $\mathfrak{F}_n$-measurable cover of A. If there exists $M > 0$ such that $PC_n(A) \leqq MPA$ for every $A \varepsilon \mathfrak{F}_{n+1}$, $n = 1, 2, \cdots$, then $(\mathfrak{F}_n, n \geqq 1)$ is said to be regular.

Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a submartingale and $E|x_n| < \infty$ for each n. If $(\mathfrak{F}_n,$

Received 3 March 1967; revised 5 September 1967.

[1] This research was supported by the National Science Foundation under Grant No. GP-06073.