The Annals of Mathematical Statistics
1968, Vol. 39, No. 1, 70–75

WHEN ARE GAUSS-MARKOV AND LEAST SQUARES ESTIMATORS IDENTICAL? A COORDINATE-FREE APPROACH[1]

BY WILLIAM KRUSKAL

University of Chicago

1. Introduction. In the framework of the general linear hypothesis, it is well known that the Gauss-Markov (minimum variance linear unbiased) and least squares estimators may be the same even when the underlying covariance structure is not a multiple of the identity. The main purpose of this note is to develop a (known) condition for this phenomenon simply in terms of the coordinate-free approach to the subject.

For general background and bibliography see Zyskind (1967) and Watson (1967). Much of the present paper is effectively a simpler form of part of the material in the Zyskind and Watson papers. For discussion of the coordinate-free approach see Kruskal (1961). Two very simple examples may be useful at the outset.

EXAMPLE 1. *One-way layout with unequal variances in known ratios.* The observations correspond to uncorrelated random variables Y_{ij} ($i = 1, \cdots, I$; $j = 1, \cdots, J_i$), $EY_{ij} = \beta_i$, and $\operatorname{Var} Y_{ij} = \lambda_i^2\sigma^2$. Here the β_i are unrestricted, unknown numbers; σ^2 is an unknown positive number; and the λ_i^2 are known positive numbers. It is readily seen that the Gauss-Markov and least square estimators of β_i are both $\bar{Y}_{i\cdot} = J_i^{-1}\sum_j Y_{ij}$. The corresponding conventional estimators of σ^2 are, however, not the same, unless all the λ_i^2 are one. These conventional estimators are $[\sum(J_i - 1)]^{-1}$ times

$$\sum_{ij}(Y_{ij} - \bar{Y}_{i\cdot})^2 \qquad \text{(for Least Squares)},$$

$$\sum_{ij}(Y_{ij} - \bar{Y}_{i\cdot})^2/\lambda_i^2 \qquad \text{(for Gauss-Markov)}.$$

(Of course if the λ_i^2 are equal, the two expressions are proportional, and if all $\lambda_i^2 = 1$, the expressions are identical. To obtain unbiased estimators of σ^2, the first expression must be divided by $\sum[(J_i - 1)\lambda_i^2]$ and the second by $\sum(J_i - 1)$.)

EXAMPLE 2. *Single sample with permutation-invariant covariance structure.* The observations correspond to random variables Y_i ($i = 1, \cdots, n$), $EY_i = \beta$, $\operatorname{Var} Y_i = \sigma^2$, and $\operatorname{Cov}(Y_i, Y_{i'}) = \sigma^2\gamma$ for $i \neq i'$. Here β is an unrestricted unknown number, σ^2 is an unknown positive number, and γ is a known number

Received 25 July 1967.

[1] This research was carried out in the Department of Statistics, University of Chicago, under partial sponsorship of the Statistics Branch and the Army Research Office, Office of Naval Research, and Air Force Office of Scientific Research by Contract No. Nonr 2121(23), NR 342-043 and in part by Research Grant No. NSF GP 6562 from the Division of Mathematical, Physical and Engineering Sciences of the National Science Foundation. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.*
www.jstor.org