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ON A NECESSARY AND SUFFICIENT CONDITION FOR ADMISSIBILITY OF ESTIMATORS WHEN STRICTLY CONVEX LOSS IS USED[1]

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1. Introduction. In this paper we consider a necessary and sufficient condition for the admissibility of estimators when strictly convex loss is used. The result is stated as Theorem 1. The sufficiency of the condition is obvious and has served as the basis of admissibility proofs in [1], [6], [11], [3], and [2]. The necessity of such a method of proof is relatively deep. The author claims no practical use of Theorem 1. He has been moved primarily by curiosity about the necessity part of the theorem together with a desire to strengthen the tools of decision theory. The results of this paper depend on Farrell [5] to which one can refer for definitions of some common terms like "Bayes" if these are not clear.

In the sequel $\mathbb{R}_n$ will denote Euclidean n-space $n \geqq 1$, and $\mathbb{R} = \mathbb{R}_1$ the set of real numbers. If $n \geqq 1$ we let $\mathcal{B}_n$ be the σ-algebra of Borel subsets of $\mathbb{R}_n$.

Theorem 1. *Let $X = \mathbb{R}_k$. Let the decision space $\mathfrak{D}$ be an open convex subset of $\mathbb{R}_m$. We suppose the parameter space Ω is a separable locally compact metric space and $\mathcal{C}$ is the σ-algebra of Borel subsets of Ω. In addition we assume*

(i) *μ is a σ-finite (regular) measure on $\mathcal{B}_k$.*

(ii) *$\{f(\cdot, \omega), \omega \varepsilon \Omega\}$ is a family of density functions in $L_1(X, \mathcal{B}_k, \mu)$.*

(iii) *$f: X \times \Omega \to [0, \infty)$ is (jointly) measurable and if $x \varepsilon X$ then $f(x, \cdot): \Omega \to [0, \infty)$ is a continuous function.*

(iv) *The measure of loss W satisfies, $W: \mathfrak{D} \times \Omega \to [0, \infty)$ is a continuous function. If $\omega \varepsilon \Omega$, $W(\cdot, \omega): \mathfrak{D} \to [0, \infty)$ is strictly convex. If $E \subset \Omega$ is compact then $\lim_{|t|\to\infty} \inf_{\omega \varepsilon E} W(t, \omega) = \infty$.*

(v) *If $\omega \varepsilon \Omega$, $x \varepsilon X$, then $f(x, \omega) > 0$.*

Then (vi) *and* (vii) *stated below are equivalent.*

(vi) *The estimator δ is admissible;*

(vii) *The procedure δ is non-randomized and has risk function $r(\delta, \cdot)$. There exists an increasing sequence of compact subsets $\{F_n, n \geqq 1\}$ of Ω, $F_n \uparrow \Omega$, a sequence of finite measures $\{\eta_n, n \geqq 1\}$, and a sequence of Bayes procedures $\{\delta_n, n \geqq 1\}$, such that*

(vii a) *there exists a compact subset $E_0 \subset \Omega$ such that $\inf_{n \geqq 1} \eta_n(E_0) \geqq 1$;*

(vii b) *if $E \subset \Omega$ is a compact subset then $\sup_{n \geqq 1} \eta_n(E) < \infty$;*

(vii c) *if $n \geqq 1$ then δ_n is Bayes relative to η_n with risk function*

$$r(\delta_n, \cdot) \cdot \mathrm{Lim}_{n\to\infty} \int (r(\delta_n, \omega) - r(\delta, \omega))\eta_n(d\omega) = 0;$$

(vii d) $\lim_{n\to\infty} r(\delta_n, \omega) = r(\delta, \omega)$, $\omega \varepsilon \Omega$.

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