The Annals of Mathematical Statistics
1968, Vol. 39, No. 2, 547–560

LIKELIHOOD RATIO TESTS FOR RESTRICTED FAMILIES OF PROBABILITY DISTRIBUTIONS[1]

BY RICHARD E. BARLOW

University of California, Berkeley

1. Introduction. Recently, a conditional likelihood ratio test has been proposed for testing for trend in a stochastic process of Poisson type [Boswell (1966)]. This is a departure from the standard literature in that the underlying family of distributions considered is essentially nonparametric. His main result is the asymptotic distribution of the likelihood ratio under the null hypothesis of no trend.

We consider likelihood ratio tests for certain geometrically restricted families of distributions. For example, let

$$\mathfrak{F}_0 = \{F \mid F(0) = 0 \text{ and } -\log [1 - F(x)]x^{-1} \text{ nondecreasing in } x \geqq 0\}.$$

Then $\mathfrak{F}_0$ is known as the IFRA (for increasing failure rate average) family of distributions. These distributions play an important role in the mathematical theory of reliability [Birnbaum, Esary, and Marshall (1966)]. However, not only is the family nonparametric but there is no sigma-finite measure relative to which all $F \varepsilon \mathfrak{F}_0$ are absolutely continuous. Hence, the usual concept of maximum likelihood estimate does not suffice. Kiefer and Wolfowitz (1956), p. 893, propose a generalization of the maximum likelihood estimate concept which we adopt. Let $F_1, F_2 \varepsilon \mathfrak{F}$ and let $f(\cdot; F_1, F_2)$ denote the Radon-Nikodym derivative of F_1 with respect to the measure induced by $F_1 + F_2$.

DEFINITION 1. $\hat{F}$ is called the *maximum likelihood estimate* relative to $\mathfrak{F}$ if $\hat{F}$ satisfies

$$\sup_{F \varepsilon \mathfrak{F}} \prod_{i=1}^{n} \{f(X_i; F, \hat{F})[1 - f(X_i; F, F)]^{-1}\} = 1,$$

where $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ is a random sample.

This definition is easily seen to coincide with the usual definition when the family $\mathfrak{F}$ is dominated by a sigma-finite measure.

Now consider the problem of testing $H_0 : F \varepsilon \mathfrak{F}_0$ against the alternative $H_1 : F \varepsilon \mathfrak{F} - \mathfrak{F}_0$ where $\mathfrak{F}_0 \subset \mathfrak{F}$. Let $\hat{F}_0(\hat{F})$ denote the maximum likelihood estimate relative to $\mathfrak{F}_0(\mathfrak{F})$ in the sense of Definition 1. We define the likelihood ratio statistic $\Lambda_n(\mathbf{X})$ based on a random sample $\mathbf{X}$ as follows:

DEFINITION 2. $\Lambda_n(\mathbf{X})$ is called the *likelihood ratio statistic* where

$$\Lambda_n(\mathbf{X}) = \prod_{i=1}^{n} \{f(X_i; \hat{F}_0, \hat{F})[1 - f(X_i; \hat{F}_0, \hat{F})]^{-1}\}.$$

We will be concerned with the properties of $\Lambda_n(\mathbf{X})$ for various restricted families

Received 12 June 1967; revised 23 October 1967.

[1] Research partially supported by the Office of Naval Research Contract Nonr-3656(18) with the University of California. Work initiated while the author was a Visiting Research Associate at Boeing Scientific Research Laboratories.