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A SIMPLER PROOF OF SMITH'S ROULETTE THEOREM[1]

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A roulette-table is governed by two parameters w and r with $0 < w < r < 1$, where: w is the probability that a player who stakes a unit amount of money on a single hole on a particular spin of the wheel will, on that particular spin, win; and $1/r$ is the number of units that the house then returns to him if he wins that bet, that is, $(1/r) - 1$ is the amount that he gains from that bet. (In many real-world casinos, w is $1/38$ and r is $1/36$.)

How should someone with an infinitely divisible fortune play so as to maximize the probability of ultimately attaining a specified larger fortune? A step toward answering this question was made in [2], Chap. 6, where it was shown that bold play is optimal if a positive stake may be placed on only one hole on each spin. The second and final step was taken by Smith in [3] where he showed that (if w and r are reciprocals of integers) there is no advantage in placing positive stakes on more than one hole. (Theorem 1 below.)

The purpose of this note is to give a shorter and simpler proof of Smith's result. Though valid for all real w and r, $0 < w < r < 1$, the proof given here is in large measure simply a reorganization of Smith's. This simplification (and slight generalization) is achieved by establishing and exploiting (7), and (7) is an immediate consequence of this inequality:

PROPOSITION 1. *For every subfair casino function* U,

$$U(f/(1-f)) \geqq U(f)/(1-U(f)) \quad \text{for} \quad 0 \leqq f \leqq \tfrac{1}{2}, \tag{1}$$

and, more generally, for each integer $n \geqq 1$,

$$U(f/(1-nf)) \geqq U(f)/(1-nU(f)) \quad \text{for} \quad 0 \leqq f \leqq 1/(n+1). \tag{2}$$

PROOF. As was shown for primitive casino functions in [2], Chapter 6, and for all subfair casino functions in [1],

$$U(f+g) \geqq U(f) + U(g) \quad \text{for} \quad 0 \leqq f+g \leqq 1. \tag{3}$$

Moreover,

$$U(fg) \geqq U(f)U(g), \tag{4}$$

as was pointed out in [2], Chapter 4.

Hence, for $0 \leqq f \leqq \frac{1}{2}$,

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