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AN INEQUALITY IN CONSTRAINED RANDOM VARIABLES

BY C. L. MALLOWS

Bell Telephone Laboratories, Incorporated

Suppose the random variables $\{X_{ij}, i = 1, \cdots, m, j = 1, \cdots, n\}$ are independent and identically distributed. Then the event A defined by the inequalities

$$X_{11} + X_{12} + \cdots + X_{1n} \leqq a_1,$$

$$X_{21} + X_{22} + \cdots + X_{2n} \leqq a_2,$$

$$X_{m1} + X_{m2} + \cdots + X_{mn} \leqq a_n$$

has a certain probability, $P(A)$. If the random variables are constrained to satisfy some conditions of the form $X_{11} = X_{21}$, $X_{23} = X_{63}$ etc. (but not of the form $X_{11} = X_{12}$ etc.), then $P(A)$ is increased. This result was conjectured by E. Arthurs.

THEOREM. *If*

(i) $X = \{X_{ij}, i = 1, \cdots, m, j = 1, \cdots, n\}$ *is an array of independent and identically distributed random variables;*

(ii) A *is an event of the form*

$$A = \{\sum_{j=1}^{n} X_{ij} \leqq a_i, i = 1, \cdots, m\};$$

(iii) C_1, C_2 *are sets of constraints of the form* $X_{ij} = X_{i'j}$ *with* $C_1 \subset C_2$*; then*

$$P(A \mid C_1) \leqq P(A \mid C_2).$$

PROOF. It is sufficient to prove the result in the case that C_1 does not imply $X_{11} = X_{21}$, while $C_2 = C_1 \cup \{X_{11} = X_{21}\}$. Suppose the dimensionality of X conditioned by C_1 is $k + 2$, where $k \geqq 0$. We can choose k variables $Y_1, Y_2, \cdots, Y_k$ out of X so that $Y_1, Y_2, \cdots, Y_k, X_{11}, X_{21}$ are linearly independent under C_1. Then $Y_1, \cdots, Y_k, X_{11}, X_{21}$ are statistically independent, and

$$P(A \mid C_1) = \int_B \prod_{i=1}^{k} dF(y_i) \int_{-\infty}^{b_1} dF(x_{11}) \int_{-\infty}^{b_2} dF(x_{21})$$

where $F(x) = P(X_{11} \leqq x)$, B is some region in R^k (which is the range of $(y_1, \cdots, y_k)$), and b_1, b_2 are certain functions of $y_1, \cdots, y_k$ derived from the inequalities defining the event A. This is to be compared with

$$P(A \mid C_2) = \int_B \prod_{i=1}^{k} dF(y_i) \int_{-\infty}^{\min(b_1, b_2)} dF(x).$$

However, for any distribution function F and any b_1, b_2

$$\int_{-\infty}^{b_1} dF(u) \int_{-\infty}^{b_2} dF(v) \leqq \int_{-\infty}^{\min(b_1, b_2)} dF(x)$$

and the theorem is proved.

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