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FUNCTIONS OF FINITE MARKOV CHAINS AND EXPONENTIAL TYPE PROCESSES

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1. Introduction and notation. When is an arbitrary random process, $Y(t)$, equal, in joint distribution, to a function of a Markov chain?

When is a function of a Markov chain, $f[X(t)]$, itself a Markov chain?

This paper is devoted to the above questions when $Y(t)$ is an exponential type process [13], p. 207, and the Markov chain, $X(t)$ is a basic Markov chain [13], p. 207. The structure of an exponential type process of order K [13], p. 208 is analyzed.[1] A necessary and sufficient condition for an exponential type process of order K to be a function of a basic Markov chain with K states (Theorem 3.1) and a necessary and sufficient condition for an exponential type process to be a Markov chain (Theorem 4.2) are established.

If $\Phi = \{i_1, i_2, \cdots, i_n\}$ is a finite sequence of states of a random process $Z(t)$ and $S = \{s_1, s_1 + s_2, \cdots, s_1 + s_2 + \cdots + s_r\}$ is a corresponding monotone sequence of times, then the pair $(\Phi; S)$ is termed a *sequence pair of length n* for the process $Z(t)$. We denote the joint probabilities by:

$$P_Z(\Phi; S) = \Pr[Z(\tau_j) = i_j \text{ for } 1 \leqq j \leqq n]$$

where $\tau_j = \sum_{i=1}^{j} s_j$.

2. The structure of an exponential type process. Let $Y(t)$ be an exponential type process of order K, with state space $\mathfrak{M} = \{1, 2, \cdots, M\}$. The joint probabilities for $Y(t)$ are given by:

$$P_Y(\Phi; S) = \mathbf{b}[\textstyle\prod_{j=1}^{n} e^{Ds_j} B(i_j)]\mathbf{c}', \tag{2.1}$$

where $\mathbf{b} = (b_k)$ is a K-vector of the form $b_1 = 1$, $b_k = 0$ or 1 for $2 \leqq k \leqq K$, $D = \text{diag}\{0 = \nu_1, \nu_2, \cdots, \nu_k\}$ is a $K \times K$ diagonal matrix, $B(m) = (b_{\alpha\beta}(m))$ for $1 \leqq m \leqq M$ are the $K \times K$ matrices appearing in the definition of exponential type, and $\mathbf{c}' = (1, 0, \cdots, 0)'$ is transpose of the K-vector $(1, 0, \cdots, 0)$.

A set of M, $K \times K$ matrices $R(1), R(2), \cdots, R(M)$ is termed a set of *factor* matrices provided that $R(m)R(m) = R(m)$ for $1 \leqq m \leqq M$, $R(k)R(m)$ is the zero matrix whenever $k \neq m$ and $\sum_{m=1}^{M} R(m) = I$, the $K \times K$ identity matrix. The first result here is that the M matrices associated with an exponential type process of order K constitute a set of factor matrices.

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[1] The results reported herein are valid under more general definitions of exponential type process and basic Markov chain than those used by Leysieffer [13]. We only require that the ν's (eigenvalues) be distinct complex numbers with non-positive real parts and $\nu_1 = 0$. Also the restriction that all initial probabilities be non-zero is replaced with the trivial requirement that the state spaces be the "essential" state space. That is, we assume that if m is a state of the process $Z(t)$ then for some $t \geqq 0$, $\Pr[Z(t) = m] > 0$.