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ON THE DISTRIBUTION OF THE MAXIMUM OF A SEMI-MARKOV PROCESS[1]

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0. Introduction. If $\{X(t): t \geqq 0\}$ is a separable stochastic process, the problem of computing the distribution of $Z(t) = \sup [X(s): 0 \leqq s \leqq t]$ is of great interest particularly in level crossing (detection) problems and in queuing theory.

Spitzer [8] used combinatorial methods to find the distribution of $Z(t)$ in the case of a discrete time random walk. In [1] Baxter used operator theoretic techniques to give a characterization of the distribution of $Z(t)$ and many other functionals on a discrete time Markov process. In the case of continuous time processes with stationary independent increments Baxter and Donsker [2] obtained the double Laplace transform of the distribution of $Z(t)$. Using a generalization of the classical ballot theorem, Takacs [9], has computed the distribution of $Z(t)$ for many interesting cases involving processes with interchangeable increments.

However, there are many cases in which one must deal with continuous time Markov processes and semi-Markov processes. The purpose of this paper is to extend the results of Baxter [1] by characterizing the distribution of $Z(t)$ for a wide class of semi-Markov processes.

Define $m_{ij}(s)$ to be the Laplace transform of the function $M_{ij}(t) = P[Z(t) = j \mid X_0 = i]$ and let $m(s) = (m_{ij}(s))$. The main result of this paper is in the form of a recurrence relation for $m(s)$

$$m(s) = g(s) + (q(s)m(s))^{\sigma}$$

where $q(s)$ and $g(s)$ are matrices whose elements are Laplace transforms of distributions which occur in the definition of the semi-Markov process and σ is an operator on matrices. Moreover, $m(s)$ is the unique solution of the above equation under a condition on the matrices $q(s)$ which guarantees that the process makes a finite number of transitions in any finite interval of time.

1. Preliminaries. First it is necessary to discuss linear operations defined on a space, $\mathcal{L}$, of bounded sequences, $\{s_i\} i \varepsilon I$, where I may be an arbitrary subset of the integers. The exact nature of $\mathcal{L}$ will depend on the state space of the semi-Markov process in question. For us, the important properties of $\mathcal{L}$ are that it is a Banach space under the supremum norm and that any bounded linear oper-

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