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ON CONTINUITY PROPERTIES OF INFINITELY DIVISIBLE DISTRIBUTION FUNCTIONS

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Let $\varphi(\xi)$ be the characteristic function of an infinitely divisible distribution function F on R_1. Suppose F has no Gaussian component. Then one has the representation

$$(1) \qquad -\log|\varphi(\xi)| = \int_{-\infty}^{\infty} (1 - \cos \xi x)\nu(dx) = 2\int_{-\infty}^{\infty} \sin^2 \tfrac{1}{2}\xi x\nu(dx)$$

where ν is the Levy measure and satisfies

$$(2) \qquad \int_{-\infty}^{\infty} x^2(1 + x^2)^{-1}\nu(dx) < \infty.$$

Continuity properties of F depend on the behavior of ν near the origin. For interesting results and references to previous work see [5]. Related information of a very refined kind is developed in [3].

1. For $0 \leqq \lambda \leqq 2$ introduce the condition

$$(C_\lambda) \qquad \int_{-1}^{1} |x|^\lambda \nu(dx) = \infty.$$

It is easily seen from (2) above that (C_2) never holds. In [2] it is shown that (C_0) is equivalent to the continuity of F. An example is given in [1] in which (C_0) holds but the corresponding F is not absolutely continuous. This example will be modified to show that for any $\lambda < 2$ (C_λ) may hold yet F not be absolutely continuous. It follows that assertion (I) [2], p. 286, and the result given as Corollaries 1–3 of Theorem 2 in [4] are erroneous.

It follows from (1) and the Riemann-Lebesgue lemma that if F is absolutely continuous

$$(3) \qquad \int_{-\infty}^{\infty} \sin^2 \xi x\nu(dx) \to \infty \quad \text{as} \quad |\xi| \to \infty.$$

Let $0 \leqq \lambda < 2$, and let c be an integer exceeding $2/(2 - \lambda)$. Let $a_j = 2^{-c^j}$, $j = 1, 2, \cdots$, and let ν be atomic with atoms of weight $a_j^{-\lambda}$ at $x = a_j$, $j = 1, 2, \cdots$. Note that ν satisfies (2) and (C_λ). Let $\xi_n = \pi a_n^{-1}$ and observe

$$\begin{aligned}\int_{-\infty}^{\infty} \sin^2 \xi_n x\nu(dx) &= \sum_{j=1}^{\infty} (\sin^2 \pi\, 2^{(c^n - c^j)})2^{\lambda c^j} \\ &= \sum_{j=n+1}^{\infty} (\sin^2 \pi\, 2^{(c^n - c^j)})2^{\lambda c^j} \leqq \pi^2 \sum_{j=n+1}^{\infty} 2^{-((2-\lambda)c^j - 2c^n)}\end{aligned}$$

and the last term tends to zero with n, so that (3) is violated.

2. Let $H(r) = \int_{|x|<r} |x|^2\nu(dx)$. Consider the condition

(D_β) There exist $c > 0$ and $r_0 > 0$ such that $H(r) > cr^\beta$ for $0 < r < r_0$.

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