The Annals of Mathematical Statistics
1968, Vol. 39, No. 4, 1350–1353

ON THE PROPORTION OF OBSERVATIONS ABOVE SAMPLE MEANS IN A BIVARIATE NORMAL DISTRIBUTION[1]

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Let (x_i, y_i) $(i = 1, 2, \cdots, n)$ be n independent observations from a bivariate normal distribution with

$$E(x_i) = E(y_i) = 0, \qquad E(x_i^2) = E(y_i^2) = 1, \qquad E(x_i y_i) = \rho, \qquad |\rho| < 1.$$

Let $\bar{x}$ and $\bar{y}$ be the sample means of x and y respectively. Suppose

$$(1) \qquad \begin{aligned} P_n &= \text{proportion of } x_1, x_2, \cdots, x_n \text{ above } \bar{x}, \\ Q_n &= \text{proportion of } y_1, y_2, \cdots, y_n \text{ above } \bar{y}. \end{aligned}$$

In this note, we shall derive the limiting bivariate distribution of (P_n, Q_n). This will be a generalization of the result obtained by David [2]. An application of this result will also be pointed out.

THEOREM. *The limiting distribution of* $(n^{\frac{1}{2}}(P_n - \frac{1}{2}), n^{\frac{1}{2}}(Q_n - \frac{1}{2}))$ *is bivariate normal with means zero and dispersion matrix*

$$(2) \qquad \begin{pmatrix} \frac{1}{4} - (2\pi)^{-1} & (\text{arc sin } \rho - \rho)(2\pi)^{-1} \\ (\text{arc sin } \rho - \rho)(2\pi)^{-1} & \frac{1}{4} - (2\pi)^{-1} \end{pmatrix}.$$

PROOF. Let t_1 and t_2 be any two real numbers. Suppose,

$$p_1(n) = \tfrac{1}{2} + n^{-\frac{1}{2}}t_1; \quad q_1(n) = \tfrac{1}{2} - n^{-\frac{1}{2}}t_1;$$
$$p_2(n) = \tfrac{1}{2} + n^{-\frac{1}{2}}t_2; \quad q_2(n) = \tfrac{1}{2} - n^{-\frac{1}{2}}t_2.$$

Let U_q and ξ_q be the sample and the population quantiles of order q for x. Let V_q and η_q be the corresponding expressions for y.

$$(3) \qquad \begin{aligned} &P\{n^{\frac{1}{2}}(P_n - \tfrac{1}{2}) \leqq t_1, n^{\frac{1}{2}}(Q_n - \tfrac{1}{2}) \leqq t_2\} \\ &= P\{P_n \leqq p_1(n), Q_n \leqq p_2(n)\} = P\{U_{q_1(n)} \leqq \bar{x}, V_{q_2(n)} \leqq \bar{y}\} \\ &= P\{n^{\frac{1}{2}}(U_{q_1(n)} - \bar{x} - \xi_{q_1(n)}) \leqq -n^{\frac{1}{2}}\xi_{q_1(n)}, \\ &\qquad n^{\frac{1}{2}}(V_{q_2(n)} - \bar{y} - \eta_{q_2(n)}) \leqq -n^{\frac{1}{2}}\eta_{q_2(n)}\}. \end{aligned}$$

Let,

$$a_n = n^{\frac{1}{2}}(U_{q_1(n)} - \bar{x} - \xi_{q_1(n)}), \quad b_n = n^{\frac{1}{2}}\bar{x}, \quad c_n = n^{\frac{1}{2}}(V_{q_2(n)} - \bar{y} - \eta_{q_2(n)}), \quad d_n = n^{\frac{1}{2}}\bar{y}.$$

Then,

$$a_n + b_n = n^{\frac{1}{2}}(U_{q_1(n)} - \xi_{q_1(n)}), \qquad c_n + d_n = n^{\frac{1}{2}}(V_{q_2(n)} - \eta_{q_2(n)}).$$

Received 11 October 1967.
[1] Research partially supported by Grant No. WP00457 of the Department of the Interior.