The Annals of Mathematical Statistics
1968, Vol. 39, No. 4, 1348–1349

A NOTE ON THE WEAK LAW OF LARGE NUMBERS

BY MELVIN KATZ[1]

State University of New York at Albany

Let $\{X_k : k \geqq 1\}$ denote a sequence of independent and identically distributed (iid) random variables. Let $S_n = \sum_{k=1}^{n} X_k$. If S_n/n converges to zero in probability but not with probability one (wp 1) it is well known that $\limsup_n |S_n|/n = +\infty$ wp 1. The purpose of this note is to show that in fact $\limsup_n S_n/n = +\infty$ wp 1 and $\liminf_n S_n/n = -\infty$ wp 1.

LEMMA. *Let* $\{X_k : k \geqq 1\}$ *be* iid *then the following are equivalent.*

(a) $\limsup_n S_n/n = +\infty$ wp 1.

(b) $\sum_{n=1}^{\infty} n^{-1} P(S_n > nM) = \infty$ *for all* $M > 0$.

PROOF. Suppose that (a) holds. Then $\limsup_n (S_n - nM)/n = +\infty$ wp 1 for all positive M. Consequently $\limsup_n (S_n - nM) = \infty$ wp 1 for all positive M. Therefore by (Theorem 4.1, [1]) (b) holds. Conversely if (b) holds it follows, again from (Theorem 4.1, [1]), that $\limsup_n (S_n - nM) = +\infty$ wp 1 for all positive M and therefore (a) holds.

LEMMA. *Let* $X_k : k \geqq 1\}$ *be* iid, $\epsilon > 0$, *and suppose that* S_n/n *converges to zero in probability. Then there exists a positive constant* A *such that*

$$P(S_n > n\epsilon) \geqq AnP(X_1 > 2n\epsilon) \quad \textit{for} \quad n \geqq n_0.$$

PROOF. Let $\mu(X)$ denote the median of the random variable X and let $S_n^{\,i} = \sum_{k=1, k \neq i}^{n} X_k$. Then

$$\begin{aligned} P(S_n > n\epsilon) &\geqq P \textstyle\bigcup_{i=1}^{n} \{[X_i > n\epsilon - (n-1)\mu(S_n^{\,i}/(n-1))] \\ &\qquad \cap [S_n^{\,i} > (n-1)\mu(S_n^{\,i}/(n-1))]\} \\ &\geqq \textstyle\sum_{i=1}^{n} [\frac{1}{2} - (i-1)P(T_1)]P(T_i) \end{aligned}$$

where $T_i = [X_i > n\epsilon - (n-1)\mu(S_n^{\,i}/(n-1))]$. Further S_n/n converging to zero in probability implies that $\mu(S_n^{\,i}/(n-1)) \rightarrow 0$ and $nP[|X_1| > n\epsilon/2] \rightarrow 0$. Therefore there exists n_0 such that if $n \geqq n_0$ it follows that

$$P(X_1 > 2n\epsilon) \leqq P(T_1) \leqq P(X_1 > n\epsilon/2) \quad \text{and} \quad [\tfrac{1}{2} - nP(X_1 > n\epsilon/2)] \geqq A > 0.$$

Thus if $n \geqq n_0$ it follows that $P(S_n > n\epsilon) \geqq AnP(X_1 > 2n\epsilon)$.

THEOREM. *Let* $\{X_k : k \geqq 1\}$ *be* iid *and suppose* S_n/n *converges to zero in probability but not* wp 1. *Then* $\limsup_n S_n/n = +\infty$ wp 1 *and* $\liminf_n S_n/n = -\infty$ wp 1.

PROOF. First note that $\sum_{n=1}^{\infty} P(X_1 > 2n\epsilon) = \infty$ for all $\epsilon > 0$. For if $\sum_{n=1}^{\infty} P(X_1 > 2n\epsilon) < \infty$ for some $\epsilon > 0$ it would follow that $EX_1^{+} < \infty$; and

Received 8 November 1967.

[1] Research partially supported by NSF Grant GP-1816.