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PROBABILITY DENSITIES WITH GIVEN MARGINALS[1]

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1. Introduction. Ireland and Kullback (1968) considered the problem of estimating contingency tables with given marginals on the basis of an observed contingency table, by minimizing a discrimination information value. It was noted that the procedures they described for the case of discrete distributions may also be extended to probability densities. It is the purpose of this paper to carry out the appropriate extension. It will be noted that although the procedures and results are developed in detail for bivariate densities, as a matter of convenience, there is nothing inherent in the techniques restricting the results to bivariate densities, and indeed in Section 3 are given appropriate results for a four-variate density.

The following is the formulation of the problem to be considered. Let $\pi(x, y)$ be some bivariate probability density, and required the bivariate probability density $f(x, y)$ with given marginal probability densities $g(x)$, $h(y)$, such that

$$I(f; \pi) = \int\int f(x, y) \ln f(x, y)/\pi(x, y)\, dx\, dy \tag{1.1}$$

is a minimum over all bivariate probability densities with the given marginals. (See the discussion in Kullback [(1959), Chapter 5] and that in Good (1963), (1966) of a principle of minimum discriminability.) Note that if $I(f; \pi) < \infty$ then $f(x, y)$ determines a probability measure which is absolutely continuous with respect to the probability measure determined by $\pi(x, y)$ [Kullback, (1959), p. 5].

In order to apply the minimum discrimination information theorem [Kullback and Khairat, (1966)] to the problem formulated above define

$$T_x(\xi, \eta) = \delta(x - \xi), \qquad T_y(\xi, \eta) = \delta(y - \eta), \tag{1.2}$$

where δ is the Dirac delta-function [Rényi, (1962), p. 298] so that

$$\int\int T_x(\xi, \eta) f(\xi, \eta)\, d\xi\, d\eta = \int\int \delta(x - \xi) f(\xi, \eta)\, d\xi\, d\eta = \int \delta(x - \xi) g(\xi)\, d\xi = g(x). \tag{1.3}$$

$$\int\int T_y(\xi, \eta) f(\xi, \eta)\, d\xi\, d\eta = \int\int \delta(y - \eta) f(\xi, \eta)\, d\xi\, d\eta = \int \delta(y - \eta) h(\eta)\, d\eta = h(y). \tag{1.4}$$

By the minimum discrimination information theorem, the minimizing function is

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