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RENEWAL THEOREMS WHEN THE FIRST OR THE SECOND MOMENT IS INFINITE[1]

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The classical renewal theorems do not tell much about the renewal function if the mean renewal lifetime is infinite.

To obtain more accurate results we prove a theorem that can be considered as the analogue of Smith's key renewal theorem [8] if $1 - F(t) \sim t^{-\alpha}L(t)$ for $t \to \infty$ where $L(t)$ is slowly varying and $0 < \alpha \leqq 1$.

In Section 3 we consider $1 < \alpha < 2$. An application of the main theorem yields a precise estimate for the renewal function in that case.

1. Regularly varying functions. In this section we collect a number of results that will be applied throughout the entire chapter. For a general discussion, see W. Feller [4].

DEFINITION 1. A function $L(t)$ is called *slowly varying* if $L(t)$ is defined for $t > 0$, positive, and if $\lim_{t\to\infty} L(xt)/L(t) = 1$ for all $x > 0$. We write $L(t)$ is sv.

DEFINITION 2. A distribution function $F(t) \in V_\alpha$ for $\alpha \geqq 0$ if there exists a slowly varying function $L(t)$ such that

$$(1) \qquad 1 - F(t) \sim t^{-\alpha}L(t) \quad \text{as} \quad t \to \infty.$$

The real number α is the *exponent* of $F(t)$, and $F(t)$ is said to be a *regularly varying distribution* with exponent α. It is easy to show that if (1) holds for some $\alpha \geqq 0$, then this α is unique.

The class V_α is a subclass of the family of regularly varying functions as defined by Feller [4], K. Knopp [6] and others. If $\alpha = 0$ then we assume that $F(t) < 1$ for every $t \geqq 0$. V_0 reduces to a class of slowly varying functions. A paper by S. Aljančić, R. Bojanič and M. Tomić [1] (later on referred to as A.B.T.) contains a number of important results, that will be used later.

LEMMA 1.

(i) *If* $L(t)$ *is* sv *and* $u > 0$ *then* $L(ut)/L(t) \to 1$ *as* $t \to \infty$ *uniformly in every finite interval*; $L(t)t^\gamma \to \infty\, (\to 0)$ *if* $\gamma > 0$ $(\gamma < 0)$;

(ii) *If* $L_1(t)$ *and* $L_2(t)$ are sv, *so are* $L_1(t)L_2(t)$ *and* $L_1(t)/L_2(t)$;

(iii) *If* $L(t)$ *is* sv *for* $t \geqq a$, *so is* $\int_a^t x^{-1}L(x)\,dx$;

The last part is due to S. Parameswaran [7] and W. L. Smith [8].

One of the main properties of sv functions is expressed in the following lemma, which combines an Abelian and Tauberian theorem. An elementary proof is given by Feller in ([4], p. 421).

LEMMA 2. *If* $L(t)$ *is* sv *and* $G(t)$ *is a positive, monotone and right hand continuous*

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