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A CLASS OF INFINITELY DIVISIBLE MIXTURES

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1. Introduction. In a previous paper [3] it was proved that mixtures of characteristic functions (cf's) of the form

$$(1) \qquad \lambda/(\lambda - it) \qquad (\lambda > 0)$$

are infinitely divisible (inf div). In this paper mixtures of cf's of the more general type

$$\text{I} \qquad \lambda/(\lambda - h(t))$$

are considered. It will be shown that mixtures of cf's of type I are inf div if $h(t)$ is such that $\lambda/(\lambda - h(t))$ is a cf for all $\lambda > 0$. The class of functions $h(t)$ satisfying this condition will be determined.

2. Preliminaries. In our proof we will make use of the Lévy-Khinchine canonical representation: $\phi(t)$ is an inf div cf if and only if

$$(2) \quad \log \phi(t) = ait + \int_{-\infty}^{\infty} \{e^{itx} - 1 - itx/(1 + x^2)\}(1 + x^2)x^{-2}\, d\theta(x),$$

where a is a real constant and $\theta(x)$ is bounded and non-decreasing (see e.g. [2], p. 89).

Further we shall need the well-known fact (cf. [2], p. 203) that a function of the type

$$\text{II} \qquad \lambda/(\lambda + 1 - g(t)) \qquad (g(t) a \text{ cf}; \lambda > 0)$$

is an inf div cf. This is easily seen by writing $\lambda^{1/n}(\lambda + 1 - g(t))^{-1/n}$ as a linear combination of cf's:

$$(3) \quad \lambda^{1/n}(\lambda + 1 - g(t))^{-1/n} = \{\lambda/(\lambda + 1)\}^{1/n} \sum_{k=0}^{\infty} \binom{-1/n}{k}(-1 - \lambda)^{-k}\{g(t)\}^k = \sum_{k=0}^{\infty} C_k^{(n)}\{g(t)\}^k,$$

where $C_k^{(n)}$ can be written as

$$(4) \quad C_k^{(n)} = n^{-1}(1 + n^{-1}) \cdots (k - 1 + n^{-1})(k!)^{-1}\lambda^{1/n}(1 + \lambda)^{-k-1/n} \quad (k \geqq 1).$$

3. Two lemmas.

LEMMA 1. *If* $p_j > 0$, $\sum_1^n p_j = 1$ *and* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$, *then*

$$\sum_{j=1}^n p_j\lambda_j/(\lambda_j - h) \doteq [\prod_{j=1}^n \lambda_j/(\lambda_j - h)] \prod_{k=1}^{n-1} (\mu_k - h)/\mu_k,$$

where $\lambda_j < \mu_j$ *for* $j = 1, 2, \cdots, n - 1$.

PROOF. See [3].

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