The Annals of Mathematical Statistics
1968, Vol. 39, No. 5, 1755–1758

ANCILLARY STATISTICS AND ESTIMATION OF THE LOSS IN ESTIMATION PROBLEMS

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Consider a stochastic variable X with probability density $f(x; \theta, \mu, \nu)$ where θ is a scalar whose value we want to estimate, and where μ and ν are nuisance (if they are present). We assume that all parameters can take on values independently of each other. Let the estimator be denoted by $\hat{\theta}$, let the loss be $L(\theta, \hat{\theta})$, and let the risk be $R = EL$.

Suppose that we have chosen the estimator $\hat{\theta}(X)$ (according to one or another principle), and now want a measure of the accuracy of the estimate $\hat{\theta}(x)$. If the loss $L(\theta, \hat{\theta}(x))$ were known, it would have been a perfect measure of the accuracy of $\hat{\theta}(x)$. But $L(\theta, \hat{\theta}(x))$ is of course unknown (except in very trivial cases). *On the other hand, one can estimate the unobservable stochastic variable* $L(\theta, \hat{\theta}(X))$ *by means of* X.

Let us denote the estimator of $L(\theta, \hat{\theta}(X))$ by $\Lambda(X)$. If $\Lambda(X)$ is close to $L(\theta, \hat{\theta}(X))$ with high probability for all values of the parameters, then it seems reasonable to consider $\Lambda(x)$ as a measure of the accuracy of $\hat{\theta}(x)$, and to present it together with $\hat{\theta}(x)$ as such a measure.

In this paper, we shall consider *best unbiased estimators of* L, defined in the following way:

DEFINITION 1. $\Lambda_0(X)$ is a *best unbiased estimator of* L if it is an unbiased estimator of R, i.e.

$$(1) \qquad E\Lambda_0(X) \equiv R \quad \text{or} \quad (E[\Lambda_0(X) - L(\theta, \hat{\theta}(X))] \equiv 0),$$

and if

$$(2) \qquad E[\Lambda_0(X) - L(\theta, \hat{\theta}(X))]^2 \leqq E[\Lambda(X) - L(\theta, \hat{\theta}(X))]^2$$

for all $\Lambda(X)$ such that (1) is satisfied.

THEOREM 1. *Suppose that* T *is a complete sufficient statistic, and that* L *depends on* X *only through* T. *Then there is at most one estimator* $\Lambda(T)$ *of* L *such that* $E\Lambda(T) \equiv R$. *If there is one, then it is a best unbiased estimator of* L.

PROOF. Exactly as for the corresponding result for Markov-estimators of parameters.

EXAMPLE 1. $X_1, \cdots, X_n$ are independent and identically normally distributed, $EX_i = \theta$, $\text{Var}\, X_i = \sigma^2$ which is known. Let $\hat{\theta} = \bar{X} = n^{-1} \sum_{i=1}^n X_i$ and $L = (\hat{\theta} - \theta)^2$. $R = \sigma^2 n^{-1}$. $\Lambda \equiv R$ is of course an unbiased estimator of R, and because of completeness, R is the best unbiased estimator of $(\hat{\theta} - \theta)^2$.

EXAMPLE 2. Let the situation be as in Example 1, except that σ^2 is unknown. Let $\Lambda = (n-1)^{-1} \sum (X_i - \bar{X})^2$. Since $E\Lambda \equiv R$, Λ is a best unbiased estimator of $(\hat{\theta} - \theta)^2$, because of completeness.

Received 27 September 1967.