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ON THE MONOTONICITY OF $E_p(S_t/t)$

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Let $S_n = X_1 + \cdots + X_n$ be the partial sums of iid random variables X_n with $P(X_n = 1) = p = 1 - P(X_n = 0)$. Let t be a stopping time relative to the sequence $\{X_n\}$. The following result was conjectured by H. Robbins: If $P_p\ (t < \infty) = 1$ for every $0 < p < 1$, then $E_p(S_t/t) \leqq E_{p'}(S_t/t)$ for $0 < p < p' < 1$. In this note we verify this result when the X_n are iid with density belonging to an exponential family, which includes the binomial, poisson and normal distributions. In the proof, Wald's equation $E_pS_t = E_pX_1E_pt$ for a bounded stopping time t is utilized.

THEOREM. *Let X_n, $n = 0, 1, \cdots$, be iid with exponential density $C(p)e^{Q(p)x}$ with respect to some σ-finite measure $d\mu$ on $(-\infty, \infty)$ where $Q(p)$ is continuous and strictly increasing on some open interval $I \subset (-\infty, \infty)$. Let $S_n = X_1 + \cdots + X_n$ and t be a stopping time such that $P_p(t < \infty) = 1$ for every $p \varepsilon I$. Then*

$$(1) \qquad E_p(S_t/t) \leqq E_{p'}(S_t/t) \quad \text{for} \quad p < p',\ p, p' \varepsilon I.$$

PROOF. Since $Q(p)$ is continuous and strictly increasing we may assume without loss of generality that $Q(p) = p$. Moreover, since $P_p(t < \infty) = 1$ and $E_pX_n^2 < \infty$, it follows that if $t_n = \min\{t, n\}$ then $\lim_{n\to\infty} E_p(S_{t_n}/t_n) = E_p(S_t/t)$. Therefore it suffices to prove (1) for bounded stopping rules t. The result is then immediate from the following lemma.

LEMMA. *If $Q(p) = p$, t is a bounded stopping rule and the conditions of the above theorem hold then*

$$(\partial/\partial p)E_p(S_t/t) = E_p(S_t - tE_pX_1)^2t^{-1}.$$

PROOF. If A_k denotes the projection of the set $[t = k]$ onto the first k coordinates then

$$E_p(S_t/t) = \sum_k k^{-1} \int_{[t=k]} S_k = \sum_k k^{-1} \int_{A_k} S_kC^k(p)e^{pS_k}\, d\mu_k$$

where $d\mu_k$ is the k-fold product of $d\mu$. Therefore since the summation is finite and differentiation under the integral is permissible (see Widder (1946), p. 240)

$$\begin{aligned}(\partial/\partial p)&E_p(S_t/t)\\ &= \sum_k k^{-1} \int_{A_k} S_k(\partial/\partial p)[C^k(p)e^{pS_k}]\, d\mu_k\\ &= \sum_k k^{-1} \int_{A_k} S_k[S_k - kE_pX_1]C^k(p)e^{pS_k}\, d\mu_k = E_pS_tt^{-1}[S_t - tE_pX_1]\\ &= E_p[S_t - tE_pX_1]^2t^{-1} + E_pX_1E_p[S_t - tE_pX_1] = E_p[S_t - tE_pX_1]^2t^{-1}\end{aligned}$$

since $E_pS_t = E_pX_1E_pt$.

REFERENCES

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