The Annals of Mathematical Statistics
1968, Vol. 39, No. 5, 1753–1754

A MIXTURE OF RECURRENT RANDOM WALKS NEED NOT BE RECURRENT

By J. Mineka

New York University

For our purposes a sequence $\{X_i\}_{i=1}^{\infty}$ of independent identically distributed random variables is said to generate a recurrent random walk if the partial sums $S_n = X_1 + X_2 + \cdots + X_n$ are recurrent in the sense that for any $\epsilon > 0$.

$$P\{|S_n| \leqq \epsilon \quad \text{for infinitely many} \quad n\} = 1.$$

The purpose of this note is to display two recurrent sequences of mutually independent identically distributed symmetric random variables $\{X_i\}$ and $\{Y_i\}$ such that

(A) if $Z_i = X_i + Y_i$, $\{Z_i\}$ is not recurrent, and

(B) if $\{W_i\}$ is a sequence of independent identically distributed random variables, independent of the first two sequences, where $P\{W_i = 1\} = P = 1 - P\{W_i = 0\}$, $P \neq 1, 0$, and $V_i = (1 - W_i)X_i + W_iY_i$, then $\{V_i\}$ is not recurrent.

Crucial to our counterexample is a result of Polya: If $\varphi(0) = 1$, $\varphi(t)$ is even, and $\varphi(t)$ is concave for $t > 0$, then $\varphi(t)$ is the Fourier transform of a probability distribution.

The classical Chung-Fuchs criterion [1] for recurrence, when applied to a sequence of i.i.d. symmetric random variables, reduces by the monotone convergence theorem to the following result:

If $\{X_i\}$ have Fourier transform $\varphi(t)$ then the partial sums of the sequence are recurrent iff for all $\delta > 0$,

$$\int_0^\delta dt/[1 - \varphi(t)] = \infty. \tag{1}$$

We construct two Fourier transforms $\varphi_1(t)$ and $\varphi_2(t)$, even and concave, satisfying condition (1), but such that

$$\int_0^\delta dt/[1 - \tfrac{1}{2}(\varphi_1(t) + \varphi_2(t))] < \infty \tag{2}$$

and also, since $\varphi_1(t)$ is monotonic for $t > 0$,

$$\int_0^\delta dt/[1 - \varphi_1(t)\varphi_2(t)] = \int_0^\delta dt/[(1 - \varphi_2(t))\varphi_1(t) + 1 - \varphi_1(t)] < \infty. \tag{3}$$

This shows that the sequences $\{Z_i\}$ and $\{V_i\}$ defined earlier are not recurrent, if $P = \frac{1}{2}$. The case $P \neq \frac{1}{2}$ is treated similarly.

Let $u(t)$ and $l(t)$ be two functions, such that

(a) $$u(0) = l(0) = 1;$$

(b) $$u(t) \quad \text{and} \quad l(t) \quad \text{are concave for} \quad t > 0;$$

Received 29 February 1968.