The Annals of Mathematical Statistics
1968, Vol. 39, No. 5, 1747–1750

ON THE ROBUSTNESS OF SOME CHARACTERIZATIONS OF THE NORMAL DISTRIBUTION

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1. Introduction. Let us introduce some definitions.

DEFINITION 1. Two distribution functions F and G are ϵ-coincident if

$$\sup_x |F(x) - G(x)| \leqq \epsilon.$$

DEFINITION 2. A distribution function F is ϵ-normal if there exist $a > 0$ and b such that

$$\sup_x |F(x) - \Phi(ax + b)| \leqq \epsilon,$$

where $\Phi(x)$ is $(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp\{-x^2/2\}\, dx$.

DEFINITION 3. Two random variables η and ζ are ϵ-independent if for every a, b, c, d, e, f

$$|\int_{ay+bz<c, dy+ez<f} dQ(y, z)| \leqq \epsilon, \tag{1}$$

where

$$Q(y, z) = P\{\eta < y, \zeta < z\} - P\{\eta < y\}P\{\zeta < z\}. \tag{2}$$

In 1956 N. A. Sapogov (Leningrad) [3] showed, that if $F_3 = F_1 * F_2$ is ϵ-normal, and if $F_1(0) = \frac{1}{2}$,

$$\int_{-N}^{N} x\, dF_1 = a_1, \qquad \int_{-N}^{N} x^2\, dF_1(x) - a_1^2 = \sigma_1^2 > 0, \qquad N = (2 \log(1/\epsilon))^{\frac{1}{2}} + 1,$$

then

$$\sup_x |F_1(x) - \Phi((x - a_1)/\sigma_1)| < C\sigma_1^{-3}(\log(1/\epsilon))^{-\frac{1}{2}}.$$

This study was continued by Hoang Hiu Nye (Moscow) [2] who showed in 1966 that, with some supplementary assumptions,

(a) ϵ-independence of the random variables of $\xi + \eta$ and $\xi - \eta$, where ξ and η are independent, implies $\beta_1(\epsilon)$-normality of the ξ and η;

(b) ϵ-independence of

$$\bar{\xi} = \sum \xi_i/n \quad \text{and} \quad S^2 = \sum (\xi_i - \bar{\xi})^2,$$

where the ξ_i are independent and have the same distribution function F, implies $\beta_2(\epsilon)$-normality of F. In his theorems the $\beta(\epsilon)$ are of the order of

$$(\log(1/\epsilon))^{-\frac{1}{2}}.$$

The purpose of this paper is to show that in some cases we can obtain a much better order of magnitude of the $\beta(\epsilon)$.

Received 19 April 1967.