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SOME MULTIVARIATE t-DISTRIBUTIONS[1]

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1. Introduction. It is our objective in this paper to consider various distributions which may be called "generalized t-distributions". Certain types of generalized t-distributions (and some of their applications) have been investigated in [1]–[8]. We have derived, under various hypotheses, many different frequency functions which could be called generalized t-distributions. However, if the distribution functions are complicated, for example, if they are represented as an infinite series of higher transcendental functions, then their practical usefulness is limited. With this in mind we have restricted ourselves to reporting only those situations where we have been able to obtain the density function in closed form.

Let $X, X_1, X_2, \cdots, X_p$ be Gaussian vectors of dimensions $p, n_1, n_2, \cdots, n_p$ respectively. Let $t_k = x_k/r_k$, $1 \leqq k \leqq p$, where $X = \{x_1, \cdots, x_p\}$ and $r_k = |X_k|$, $1 \leqq k \leqq p$, is the norm of X_k. Then $T = \{t_1, \cdots, t_p\}$ will be called a *generalized p-dimensional t random vector*. Of course, in general, $X, X_1, X_2, \cdots, X_p$ will be correlated. Many "generalized t variates" we have observed in the literature may be subsumed under the above definition.

2. Multivariate t-distributions. In the following theorems we shall use the standard notation $\Gamma(\cdot)$, $B(\cdot, \cdot)$, $K(\cdot)$, $E(\cdot)$, $D_{-\nu}(\cdot)$, ${}_1F_1(\cdot, \cdot; \cdot)$, ${}_2F_1(\cdot, \cdot, \cdot; \cdot)$ to denote, respectively, the gamma function, the beta function, the complete elliptic integral of the first kind, the complete elliptic integral of the second kind, the parabolic cylinder function, the confluent hypergeometric function, and the hypergeometric function. If Σ is a square matrix, we shall write $|\Sigma|$ to indicate the determinant of Σ. Also, if $Y = \{y_1, \cdots, y_n\}$ and $Z = \{z_1, \cdots, z_n\}$ are n-dimensional random vectors with $\{y_k, z_k\}$, $1 \leqq k \leqq n$, independent and identically distributed normal $N(0, M)$, then we shall say $\{Y, Z\}$ is of class $N_n(M)$.

Our main technique is simple. We write the density function of our t random vector as a multiple integral and apply our knowledge of special functions (see, for example, [9]) to reduce the integrals to closed form. We shall merely sketch the proofs. In particular, we shall have a number of occasions to consider various special cases of the integral:

$$J(a, b, c, n, m) \equiv \int_0^\infty \int_0^\infty x^n y^m e^{-\frac{1}{2}(ax^2+by^2)} e^{-cxy}\, dx\, dy. \tag{2.1}$$

The results we need are expressed in the following lemma.

Lemma. *Let a and b be positive, c a real number, and n a nonnegative integer.*

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