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DISTANCES OF PROBABILITY MEASURES AND RANDOM VARIABLES

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1. Introduction. Let (S, d) be a separable metric space. Let $\mathcal{P}(S)$ be the set of Borel probability measures on S. $\mathcal{C}(S)$ denotes the Banach space of bounded continuous real-valued functions on S, with norm

$$\|f\|_\infty = \sup \{|f(x)| : x \in S\}.$$

On $\mathcal{P}(S)$ we put the usual weak-star topology TW^*, the weakest such that

$$P \to \int f \, dP, \quad P \in \mathcal{P}(S)$$

is continuous for each $f \in \mathcal{C}(S)$.

It is known ([8], [11], [1]) that TW^* on $\mathcal{P}(S)$ is metrizable. The main purpose of this paper is to discuss and compare various metrics and uniformities on $\mathcal{P}(S)$ which yield the topology TW^*.

For S complete, V. Strassen [10] proved the striking and important result that if $\mu, \nu \in \mathcal{P}(S)$, the Prokhorov distance $\rho(\mu, \nu)$ is exactly the minimum distance "in probability" between random variables distributed according to μ and ν. Theorems 1 and 2 of this paper extend Strassen's result to the case where S is measurable in its completion, and, with "minimum" replaced by "infimum", to an arbitrary separable metric space S. We use the finite combinatorial "marriage lemma" at the crucial step in the proof rather than the separation of convex sets (Hahn-Banach theorem) as in [10]. This offers the possibility of a constructive method of finding random variables as close as possible with the given distributions.

For S complete, V. Skorokhod ([9], Theorem 3.1.1, p. 281) proved the related result that if $\mu_n \to \mu_0$ for TW^* there exist random variables X_n with distributions μ_n such that $X_n \to X_0$ almost surely. This is proved in Section 3 below for a general separable S. Note that it is not sufficient to establish consistent finite-dimensional joint distributions for the X_n ; the Kolmogorov existence theorem for stochastic processes is not available in this generality. Instead we construct the joint distribution of $\{X_n\}_{n=0}^\infty$ out of suitable infinite Cartesian product measures.

When S is the real line R, various special constructions involving distribution and characteristic functions are known. In Section 4, we compare some of these uniformities on $\mathcal{P}(R)$.

2. Strassen's theorem. The metric of Prokhorov [8] is defined as follows. For any $x \in S$ and $T \subset S$ let

$$d(x, T) = \inf (d(x, y) : y \in T),$$

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