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A NOTE ON THE WEAK LAW[1]

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Let $S_n = \sum_{k=1}^{n} X_k$, where X_k, $k \geqq 1$, are independent and identically distributed random variables with characteristic function φ. The object of this note is to prove the following generalization of weak law of large numbers.

THEOREM. *Let* $\alpha \in (\frac{1}{2}, \infty)$. *Then, the following are equivalent.*

(i) *There exist constants* a_n *such that* $S_n/n^{\alpha} - a_n \rightarrow_P 0$.

(ii) $|\log|\varphi(t)||^{\alpha}$ *is differentiable at* $t = 0$.

(iii) $\lim_{t\to\infty} tP\{|X_1| > t^{\alpha}\} = 0$.

PROOF. Note first that any of the conditions (i), (ii), (iii) holds for the sequence $\{X_n\}$ if and only if it holds for the symmetrized sequence $\{\hat{X}_n\}$. This is obvious for (ii). For (i) and (iii) it is an easy consequence of the symmetrization inequalities [1], p. 245. Also, if X_1 is symmetric we can take the centering constants a_n to be zero in (i). Therefore, we shall assume hereafter that X_1 is symmetric and φ is real.

First, we establish the equivalence of (i) and (ii). The logarithm of φ, as usual, is defined in a neighborhood of $t = 0$. We have $S_n/n^{\alpha} \rightarrow_P 0 \Leftrightarrow \varphi^n(t/n^{\alpha}) \rightarrow 1$, all $t \Leftrightarrow n \log \varphi(t/n^{\alpha}) \rightarrow 0$, all $t \Leftrightarrow \log \varphi(t/n^{\alpha})/|t|^{1/\alpha}/n \rightarrow 0$, all $t \neq 0 \Leftrightarrow |\log \varphi(t/n^{\alpha})|^{\alpha}/|t|/n^{\alpha} \rightarrow 0$, all $t \neq 0 \Leftrightarrow |\log \varphi(t)|^{\alpha}$ is differentiable at $t = 0$. For the last equivalence make use of the fact that the convergence is uniform on bounded intervals.

It remains to show that (i) and (iii) are equivalent. We shall essentially follow [2], p. 232. To show (iii) implies (i), let τ_n be the truncation at $\pm n^{\alpha}$. Define $T_n = \sum_{j=1}^{n} \tau_n(X_j)$. Then (iii) implies that $\lim_{n\to\infty} P(S_n \neq T_n) = 0$, and consequently it is enough to show $T_n/n^{\alpha} \rightarrow_P 0$. We have

$$P\{|T_n| > \epsilon n^{\alpha}\} \leqq E|T_n|^2 \epsilon^{-2} n^{-(2\alpha)} = \epsilon^{-2} n^{-(2\alpha-1)} \int_0^n t^{2\alpha}\, dG(t)$$
$$\leqq 2\alpha\, \epsilon^{-2} n^{-(2\alpha-1)} \int_0^n t^{2\alpha-1} [1 - G(t)]\, dt,$$

where $G(t) = P\{|X_1|^{1/\alpha} \leqq t\}$. By (iii) $\lim_{t\to\infty} t[1 - G(t)] = 0$, and this is easily seen to imply that the right hand expression has limit zero for each $\epsilon > 0$. (i) implies (iii) because

$$P\{|S_n| > n^{\alpha}\} \geqq \tfrac{1}{2} P\{\max_{1\leqq j\leqq n} |S_j| > n^{\alpha}\}$$
$$= \tfrac{1}{2}(1 - P\{\max_{1\leqq j\leqq n} |S_j| \leqq n^{\alpha}\})$$
$$\geqq \tfrac{1}{2}(1 - P\{\max_{1\leqq j\leqq n} |X_j| \leqq 2n^{\alpha}\})$$
$$\geqq \tfrac{1}{2}(1 - \exp(-n\, P\{|X_1| > 2n^{\alpha}\})).$$

This completes the proof of the theorem.

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