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A GENERALIZATION OF ITO'S THEOREM CONCERNING THE POINTWISE ERGODIC THEOREM

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1. Introduction. Let $(X, \mathfrak{F}, \lambda)$ be a σ-finite measure space. Let $L_1(\lambda) = L_1(X, \mathfrak{F}, \lambda)$ be the real Banach space of λ-integrable real functions and $L_\infty(\lambda)$ the dual of $L_1(\lambda)$. All subsets of X discussed in this paper are elements of $\mathfrak{F}$. For two sets A and B, $A \subset B$, $A = B$ mean that $\lambda(A - B) = 0$, $\lambda(A \triangle B) = 0$, respectively. All functions on X are $\mathfrak{F}$-measurable real functions and will always be considered up to λ-equivalence. For two functions f and g on X, $f = g$, $f \leqq g$ mean that the equality and the inequality, respectively, are satisfied in the almost everywhere (a.e.) sense with respect to λ. $\{f \geqq g\}$ denotes the set $\{x \mid f(x) \geqq g(x)\}$. For any set A, A' denotes its complement and 1_A designates the characteristic function of A.

Let $T:f \rightarrow fT$ be a positive linear contraction, (i.e., $\|T\|_1 \leqq 1$) on $L_1(\lambda)$ to $L_1(\lambda)$. We call T a *Markov operator* on $L_1(\lambda)$. The adjoint of T which acts on $L_\infty(\lambda)$ will be denoted by T, but we will write Tg for $g \varepsilon L_\infty(\lambda)$. The adjoint T is characterized by (1) T is a positive linear operator, (2) $T1 \leqq 1$, (3) $g_k \downarrow 0$ implies $Tg_k \downarrow 0$ ([8], p. 86). We have then $\int fT \cdot g\lambda(dx) = \int f \cdot Tg\lambda(dx)$ for $f \varepsilon L_1(\lambda)$, $g \varepsilon L_\infty(\lambda)$.

The purpose of this paper is to prove the following generalization of Ito's results ([6], Theorem 1, Lemma 2).

Theorem. *Let T be a Markov operator on $L_1(\lambda)$. Suppose that the sequence $\{(1/n) \sum_{k=0}^{n-1} wT^k \mid n = 1, 2, \cdots\}$ is weakly sequentially compact for some $w \varepsilon L_1(\lambda)$ such that $w > 0$. Then the following assertions hold:*

Assertion 1. (the pointwise ergodic theorem). For each $f \varepsilon L_1(\lambda)$,

$$\lim_{n\rightarrow\infty} (1/n) \sum_{k=0}^{n-1} fT^k \qquad \textit{exists } (\lambda\text{-a.e.}).$$

Assertion 2. (the $L_1(\lambda)$-mean ergodic theorem). For each $f \varepsilon L_1(\lambda)$,

$$\lim_{n\rightarrow\infty}(1/n) \sum_{k=0}^{n-1} fT^k \qquad \textit{exists in the } L_1(\lambda)\textit{-norm.}$$

We will prove Assertions 1 and 2 in Section 2 and 3, respectively. Certain relevant facts are stated in Section 2.

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2. Proof of Assertion 1. The following lemma follows readily from the mean ergodic theorem of Yosida and Kakutani ([7], p. 441; [11], p. 192).

Lemma 1. *If the sequence $\{(1/n) \sum_{k=0}^{n-1} wT^k\}$ is weakly sequentially compact, then the sequence converges in the $L_1(\lambda)$-norm to a function $u \varepsilon L_1^+(\lambda)$ which is invariant under T*, i.e., $uT = u$.

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