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COMPARISON TESTS FOR THE CONVERGENCE OF MARTINGALES

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1. Introduction. If $f = (f_1, f_2, \cdots)$ is a sequence of real valued functions on a probability space and $d_1 = f_1$, $d_i = f_i - f_{i-1}$, $i > 1$, let

$$f_n^* = \max(|f_1|, \cdots, |f_n|), \qquad f^* = \sup_n f_n^*,$$
$$S_n(f) = (\textstyle\sum_1^n d_i^2)^{\frac{1}{2}}, \quad \text{and} \quad S(f) = S_\infty(f) = \sup_n S_n(f).$$

In [1], Burkholder proved that if f and g are martingales relative to the same sequence of σ-fields, f is L^1 bounded, and $S_n(g) \leqq S_n(f)$, $n \geqq 1$, then g converges almost everywhere. It will be shown here that the condition $S_n(g) \leqq S_n(f)$, $n \geqq 1$, can be replaced by the weaker condition $S(g) \leqq S(f)$. Using this it requires almost no alteration of Burkholder's proofs to make the same replacement in Theorems 6 and 7 of [1].

Using essentially the same method, a theorem will be proved for L^1-bounded martingales f which gives among other things the convergence of g and finiteness of $S(g)$ if, in place of $S(g) \leqq S(f)$, we have $g^* \leqq f^*$.

2. Comparison tests for martingale convergence. Suppose g is a martingale such that if $\epsilon > 0$ then there is a stopping time t such that $P(t < \infty) < \epsilon$ and $E(S_t(g)) < \infty$. Then g converges almost everywhere by Theorem 2 of [1], which states that if f is a martingale and $E(S(f)) < \infty$ then f converges almost everywhere, since by this theorem g stopped at t will converge almost everywhere and the probability of stopping at a finite time is arbitrarily small.

Lemma 1. *If* $(f_n, \mathfrak{A}_n, n \geqq 1)$ *is a nonnegative martingale with difference sequence* $(d_n, n \geqq 1)$, *and* $\lambda > 0$, *then almost everywhere*

$$P([f_n^2 + d_{n+1}^2 + \cdots]^{\frac{1}{2}} > \lambda f_n \mid \mathfrak{A}_n) \leqq M/\lambda \tag{1}$$

where M *is the constant appearing in Theorem* 8 *of* [1], *and almost everywhere*

$$P(\sup[f_n, f_{n+1}, f_{n+2}, \cdots] > \lambda f_n \mid \mathfrak{A}_n) \leqq 1/\lambda. \tag{2}$$

Proof. Let $\lambda > 0$, n be a positive integer, $A \in \mathfrak{A}_n$ and $\alpha > 0$. Then

$$P([f_n^2 + d_{n+1}^2 + \cdots]^{\frac{1}{2}} > \lambda[f_n + \alpha], A) = P([(f_n I_A/[f_n + \alpha])^2 + (d_{n+1} I_A/[f_n + \alpha])^2 + \cdots]^{\frac{1}{2}} > \lambda) \leqq (M/\lambda) P(A),$$

using the fact that the partial sums of the series $f_n I_A/[f_n + \alpha] + d_{n+1} I_A/[f_n + \alpha] + \cdots$ form a nonnegative martingale with the L^1 norm of each partial sum equal to $E(f_n I_A/[f_n + \alpha]) \leqq E(I_A) = P(A)$, together with Theorem 8 of [1].

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