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NECESSARY CONDITIONS FOR ALMOST SURE EXTINCTION OF A BRANCHING PROCESS WITH RANDOM ENVIRONMENT[1]

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1. Introduction. In this note we shall be considering what may be called a *branching process with random environment* (BPRE); it will be a sequence $\{Z_n\}$ of random variables forming a particular sort of Markov chain.

Let $\{\zeta_n\}$ be a sequence of independent and identically distributed random variables, to be known as *environment variables*, taking values in some (possibly abstract) space Ξ. Suppose that every point $\zeta \varepsilon \Xi$ has associated with it a probability generating function $\phi_\zeta(s)$, $0 \leqq s \leqq 1$, of an integer-values random variable.

The BPRE develops as follows: $Z_0 = 1$; Z_{n+1} is the total number of offspring resulting from Z_n parents, each such parent having a random number of offspring governed by the pgf $\phi_{\zeta_{n+1}}(s)$ independently of other parents and of the environment variables other than ζ_{n+1}. This model is clearly the same as the classical Galton-Watson branching process except that we allow family-size distributions to vary stochastically from generation to generation. However, all families of a given generation are governed by the same distribution of family size. Thus the separate family trees springing from different parents in a given generation, which are independent in the classical Galton-Watson process, a fact which renders the classical Galton-Watson process so tractable, are *dependent* in the BPRE.

The BPRE will be discussed more fully elsewhere (Smith and Wilkinson, (1967)). In the present note we are concerned with proving one theorem which, taken with Theorem A below (whose proof will be given in the aforementioned reference), settles at an acceptable level of generality the question: under what circumstances will the BPRE almost surely become extinct?

At this point we need some definitions. We shall suppose the means of the family size distributions

$$\xi_n =_{\text{def}} \lim_{s \uparrow 1} (1 - \phi_{\zeta_{n+1}}(s))(1 - s)^{-1}, \qquad n = 1, 2, \cdots,$$

are independent and identically distributed proper random variables (i.e. $P\{\xi_n < \infty\} = 1$). We shall also write

$$\eta_n =_{\text{def}} \phi_{\zeta_{n+1}}(0)$$

for the probability that a parent of the nth generation shall have no offspring. We shall additionally suppose that $\{\eta_n\}$ constitutes a sequence of independent and identically distributed random variables. To avoid triviality we shall assume

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