The Annals of Mathematical Statistics
1968, Vol. 39, No. 6, 2131–2135

REPRESENTING FINITELY ADDITIVE INVARIANT PROBABILITIES[1]

By Richard A. Olshen

Stanford University

1. Introduction. Hewitt and Savage [6] have shown that finitely additive exchangeable probabilities on a product space are integral averages of power product probabilities. They prove this result as a corollary to their theorems on the countably additive case. This note adapts their technique to the study of more general invariant probabilities. From results of Farrell [4] and Choquet and Feldman ([7], Section 10) it is concluded that finitely additive invariant probabilities are averages of finitely additive ergodic probabilities.

In a countably additive context it seems necessary to impose restrictions on the Borel field being studied and on the maps used to define invariance and ergodicity. Relaxing the assumptions of one type must be balanced by strengthening those of the other (in addition to [4] and [7], see [1] and [12]). Here, however, the field of sets can be arbitrary, and the maps are assumed only to be measurable. Rather than state a host of theorems which can be proved, one particular case is proved in detail. Later on it is explained how the techniques can be applied to other problems. Several definitions of ergodicity are proposed and related to the one used. The final section contains a subjective probability interpretation of invariance and ergodicity.

2. A representation theorem. A homomorphism from one field of sets to another is a map which preserves finite unions, finite intersections, and complements. The notions of isomorphism and automorphism are defined in the obvious ways. A σ-homomorphism (isomorphism, automorphism) in addition preserves countable unions and intersections.

Assume Ω is a set, $\mathfrak{F}$ a field of subsets of Ω, and T a 1-1 bi-$\mathfrak{F}$-measurable map of Ω onto Ω. T and its powers can be viewed as automorphisms of $\mathfrak{F}$. A finitely additive probability μ on $\mathfrak{F}$ is said to be invariant if $\mu(A) = \mu(T^{-1}A)[= \mu(T^nA)$, $n = \pm 1, \pm 2 \cdots]$ for all $A \in \mathfrak{F}$; μ is ergodic if there do not exist $\delta > 0$ and $A_1, A_2, \cdots \in \mathfrak{F}$ for which $\delta < \mu(A_n) < 1 - \delta$, $\lim_{m,n\to\infty} \mu(A_n \triangle A_m) = 0$, and $\mu(A_n \triangle T^{-1}A_n) \to 0$ ($\triangle$ denotes symmetric difference). Let $\mathcal{I}(\mathfrak{F})[\mathcal{E}(\mathfrak{F})]$ be the set of finitely additive invariant [ergodic] probabilities on $\mathfrak{F}$ and let $\mathcal{B}$ be the smallest σ-field of subsets of $\mathcal{E}(\mathfrak{F})$ containing all sets of the form $\{\nu \mid \nu \in \mathcal{E}(\mathfrak{F}), \nu(A) \leqq \alpha\}$, where $0 \leqq \alpha \leqq 1$, and A is a fixed set in $\mathfrak{F}$.

(1) Theorem. *$\mathcal{I}(\mathfrak{F})$ is not empty. For each $\mu \in \mathcal{I}(\mathfrak{F})$ there is a unique countably additive probability λ on $\mathcal{B}$ satisfying*

$$\mu(A) = \int_{\mathcal{E}(\mathfrak{F})} \nu(A)\, d\lambda(\nu) \tag{2}$$

Received 11 March 1968.

[1] Prepared with partial support of U.S. Army Research Office Grant DA-ARO(D)-31-124-G726.