The Annals of Mathematical Statistics
1968, Vol. 39, No. 6, 2123–2127

ON THE LOCAL BEHAVIOR OF MARKOV TRANSITION PROBABILITIES[1]

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1. Introduction. Let $P(t) = P(t, i, j)$ be a semigroup of stochastic matrices on the countable set $I = \{i, j, \cdots\}$. Suppose

(1) $$\lim_{t\to 0} P(t, i, i) = 1 \quad \text{for each} \quad i \varepsilon I.$$

Fix one state $a \varepsilon I$ and abbreviate

$$f(t) = P(t, a, a).$$

(2) THEOREM. *Suppose* $0 < \epsilon < 1$ *and* $f(1) \leqq 1 - \epsilon$. *Then*

$$\int_0^1 f(t)\, dt < 1 - \tfrac{1}{2}\epsilon.$$

(3) THEOREM. *Suppose* $0 < \epsilon < \frac{1}{4}$ *and* $f(1) \geqq 1 - \epsilon$. *Then for all* t *in* $[0, 1]$,

$$f(t) \geqq [1 + (1 - 4\epsilon)^{\frac{1}{2}}]/2 = 1 - \epsilon - \epsilon^2 - O(\epsilon^3).$$

COROLLARY. *Suppose* $0 < \epsilon < \frac{1}{4}$ *and* $\int_0^1 f(t)\, dt \geqq 1 - \epsilon$. *Then* $f(t) > 1 - 2\epsilon$ *for* $0 \leqq t \leqq 1$.

NOTE. (3) can be restated (using algebra) in this more attractive form: if $0 < \delta < \frac{1}{2}$ and $f(1) \geqq 1 - \delta + \delta^2$, then $f(t) \geqq 1 - \delta$ for $0 \leqq t \leqq 1$.

Perhaps it is worth noting explicitly that the bounds in (2) and (3) hold for all stochastic semigroups satisfying (1). The bounds in (3) are not supposed to be sharp, but they cannot be improved much, as (4) and (5) show.

(4) EXAMPLE. For any $\delta > 0$, there is an f with

$$f(t) \leqq \delta \quad \text{for} \quad \delta \leqq t \leqq 1 - \delta \quad \text{and} \quad f(1) > e^{-1}.$$

The right coefficient for ϵ^2 in (3) is not known to us, but

(5) EXAMPLE. For $K < \frac{3}{8}$ and small $\epsilon > 0$, there is an f with

$$f(\tfrac{2}{3}) < 1 - \epsilon - K\epsilon^2 \quad \text{and} \quad f(1) > 1 - \epsilon.$$

The constant $\frac{1}{2}$ in (2) is sharp, as shown by

(6) EXAMPLE. For $K < 2$, there is an f with

$$1 - f(1) > K[1 - \int_0^1 f(t)\, dt].$$

2. Proof of Theorem (2). Sippose a is not absorbing, so $0 < f(t) < 1$ for all $t > 0$. Suppose

(7) $$f(1) = 1 - \delta.$$

Received 19 February 1968.
[1] Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant 1312-67.