The Annals of Mathematical Statistics
1968, Vol. 39, No. 6, 2098–2102

ON RECENT THEOREMS CONCERNING THE SUPERCRITICAL GALTON-WATSON PROCESS

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1. Introduction. We consider a Galton-Watson process $\{Z_n ; n = 0,1,2,\cdots\}$ initiated by a single ancestor, whose offspring distribution has probability generating function $F(s) = \sum_{j=0}^{\infty} s^j P[Z_1 = j]$, $s \varepsilon [0, 1]$, and $P[Z_1 = j] \neq 1$ for any $j = 0, 1, 2, \cdots$. In the present note, we are concerned only with the supercritical case, when $1 < m \equiv E[Z_1] < \infty$, in which case it is well known that the probability of extinction, q, is the unique real number in $[0, 1)$ satisfying $F(q) = q$. We recall that the generating function, $F_n(s)$, of Z_n is the nth functional iterate of $F(s)$ for the Galton-Watson process in general, and in the supercritical case $F_n(s) \to q$ as $n \to \infty$ for $s \varepsilon [0, 1)$. In particular $F_n(0) \uparrow q$.

Recently, a considerable amount of research has been devoted to refinements of the classical theorem concerning the convergence of the random variables (Z_n/m^n), $n = 0, 1, 2, \cdots$, (for a history of the theorem prior to these, see Harris [1]). In particular, an ultimate form of the theorem has been obtained by Kesten and Stigum [2], [6], who prove that these random variables converge a.e. to a random variable W, for which $P[W = 0] = q$ or 1, and which has a continuous density on the set of positive real numbers. Moreover $E[Z_1 \log Z_1] < \infty \Leftrightarrow P[W = 0] = q \Leftrightarrow E[W] = 1$.

It therefore follows that $E[Z_1 \log Z_1] = \infty \Leftrightarrow P[W = 0] = 1$.

Thus while Kesten and Stigum have provided a complete answer for the classical norming, by m^n, of the random variables Z_n, the limit r.v. may still be degenerate at the origin. This leads us to ask whether there exists a sequence of constants, c_n, such that (Z_n/c_n) *always* converge, in some sense, to a proper non-degenerate r.v.

We provide a partial answer to this question by producing such a sequence, for which the variables (Z_n/c_n) converge *in distribution* to such a limit variable W, for which $P[W = 0] = q$. Moreover $E[Z_1 \log Z_1] < \infty \Leftrightarrow E[W] < \infty \Leftrightarrow c_n \sim$ const m^n.

It is also shown that in this situation the random variables (Z_n/c_n) form a submartingale, although this does not appear sufficient to assert a.e. convergence.

2. Preliminary considerations. It turns out that it is relevant to use, instead of the generating function $F(s)$, the function

$$k(s) = -\log F(e^{-s}), \qquad s \geqq 0,$$

which we shall call the cumulant generating function (cgf) of Z_1. It is readily checked that the cgf of Z_n, $k_n(s)$ i.e.

Received 9 November 1967.