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# CONVERGENCE RATES FOR PROBABILITIES OF MODERATE DEVIATIONS[1]

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**1. Introduction.** In [15] Rubin and Sethuraman consider sequences $\{X_n\}$ of independent random variables with partial sums $S_n = \sum_{k=1}^{n} X_k$. Under appropriate moment conditions on the $X_k$ asymptotic forms are determined for $P[S_n > c(n \lg n)^{\frac{1}{2}}]$. As in this paper, they also study the two sided deviation problem, $P[|S_n| > c(n \lg n)^{\frac{1}{2}}]$, and term expressions of the above forms, probabilities of moderate deviations.

In this paper, the convergence rate problem is studied for both

$$P[|S_n| > c(n \lg n)^{\frac{1}{2}}] \quad \text{and} \quad P[\sup_{k \geqq n} |S_k(k \lg k)^{-\frac{1}{2}}| > c].$$

For random variables with mean zero and finite variance, it follows from the central limit theorem that $P[|S_n| > c(n \lg n)^{\frac{1}{2}}]$ tends to zero. Theorem 1 of this paper completely solves the convergence rate problem for probabilities of moderate deviations under the preceding moment conditions. Theorems 2 and 3 result from a study of the convergence rate problem when, as in [15], moments higher than the second are assumed finite.

In the last section some of the properties of moderate deviations are abstracted and analogous theorems presented in a somewhat more general setting. Here the proofs are merely outlined when similar to earlier arguments.

Throughout this paper sequences $\{X_n\}$ of independent identically distributed random variables with common distribution function $F$ are considered. A median for the random variable $X$ is denoted by $\mu(X)$, and $\Phi(x)$ represents the standard normal distribution function. $[x]$ stands for the largest integer less than or equal to $x$ and $\lg x$ is the function defined by

$$\begin{aligned} \lg x &= \log_e x \quad \text{for} \quad x > 1 \\ &= 0 \qquad\quad \text{otherwise.} \end{aligned}$$

Also positive constants are written $c$, with or without subscripts.

**2. Moderate deviations.** This section deals exclusively with probabilities of the form $P[|S_n| > c(n \lg n)^{\frac{1}{2}}]$ and $P[\sup_{k \geqq n}|S_k(k \lg k)^{-\frac{1}{2}}| > c]$ for sequences of independent identically distributed random variables. In Theorem 1 characterizations are given for such sequences with $EX_1 = 0$, $EX_1^2 < \infty$ in terms of each of the above probabilities. Lemmas 1 and 2 are useful here and in later arguments.

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