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A TREATMENT OF TIES IN PAIRED COMPARISONS[1]

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1. Introduction. This paper extends the results of Thompson and Remage (1964) and Remage and Thompson (1966) to cover the treatment of ties in obtaining maximum likelihood paired comparison rankings.

An easily understood, nonmathematical description of this problem is the following: Given the win-loss-tie records of the various major college football teams in the nation, but ignoring the scores of the games, under minimal assumptions determine the national rankings of the teams. In particular, since all teams do not play one another, determine how pairs of teams may be ranked indirectly and which pairs cannot be ranked at all.

More important scientific motivations for the research arise in psychology and other social sciences. For various sound experimental reasons the statistician will frequently encounter paired comparison data. The members of a set X of m objects are compared two at a time by a "subject" who states his preferences and indifferences. In this way two basic comparisons $x_i \rightarrow x_j$ and $x_k — x_l$, read "x_i preferred to x_j" and "x_k ties x_l", are established among the objects of X. If these objects stand in some order then as a minimum we expect that the ordering relation will be transitive and asymmetric. Such a relation is called a preference. In general, the basic comparisons will not immediately yield a preference relation; some pairs of objects will not have been compared directly and an indirect method of comparison will have to be found. Indirect comparison is the topic of Section 2. But indirect comparisons will usually be self-contradictory and some basic comparisons will have to be deleted or changed in order to obtain a preference. The effect on ranking of changing the orientation of lines is studied in Section 3.

In Section 4, to find a criterion for altering basic comparisons, we advance the following stochastic model for paired comparison data with ties. The m objects of X are independently (in the probability sense) compared in pairs. x_i and x_j are compared on $n_{ij} \geqq 0$ independent trials; each trial having three possible outcomes denoted respectively by $x_i \rightarrow x_j$, $x_j \rightarrow x_i$, and $x_i — x_j$. Let $\mathcal{I}$ denote the set of subscript pairs (ij) such that $1 \leqq i < j \leqq m$ and $n_{ij} > 0$ and let n be the number of pairs in $\mathcal{I}$. We have $n \leqq \binom{m}{2}$ with equality holding in the important case where every pair of objects is compared at least once. We consider these basic paired comparisons to constitute a sample and we introduce the population parameters $\pi_{ij} = P(x_i \rightarrow x_j)$ and $\gamma_{ij} = P(x_i — x_j)$. The probability

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