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OPTIMAL STOPPING FOR FUNCTIONS OF MARKOV CHAINS[1]

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1. The Introduction. The purpose of this paper is to prove the existence of finite optimal stopping rules for certain problems (Theorem 1 and Theorem 2), that are generalizations of a problem introduced by Y. S. Chow and H. Robbins [1] and subsequently generalized by A. Dvoretzky [3].

The problem of Y. S. Chow and H. Robbins is stated as follows: let S_n be the excess of the number of heads over the number of tails in the first n tosses of a fair coin. Does there exist a finite stopping rule for which the expected average gain is maximal? They proved the existence of such a stopping rule; subsequently A. Dvoretzky considered a sequence $X_1, X_2, \cdots$, of independent identically distributed random variables with finite variance, and proved the existence of a finite stopping rule which maximizes $E(S_t/t)$ where $S_n = X_1 + X_2 + \cdots + X_n$. Our method of proof consists of looking at the rate at which the expected tail-income $\sup_{t \varepsilon T_\infty} E(S_t/(a + t))$ goes to zero as $a \to \infty$ (where T_∞ is the class of all stopping rules). Then we use this information to show that there is an improvement for any stopping rule which continues indefinitely with positive probability.

2. Definitions and preliminaries. Let $\{X_n, F_n, n = 1, 2, \cdots\}$ be a stochastic sequence defined on a probability space (Ω, F, P), (i.e., (F_n) is an increasing sequence of sub-sigma-algebras of F, and for each $n \geqq 1$, X_n is a random variable measurable F_n), with $E|X_n| < \infty$ for $n = 1, 2, \cdots$, and $E(\sup_n X_n^+) < \infty$. Let T_∞ = class of all stopping rules with respect to (F_n), i.e., class of all $t:\Omega \to \{1, 2, \cdots, \infty\}$ such that $[t = k] \varepsilon F_k$ for $k = 1, 2, \cdots$. $T = \{t \varepsilon T_\infty : t < \infty \text{ a.s.}\}$. Given a $\tau \varepsilon T$ let,

$$T_\infty^{(\tau)} = \text{class of all random variables } t:\Omega \to \{0, 1, \cdots, \infty\}$$

such that $[t = k] \varepsilon F_{\tau+k}$.

If $t \varepsilon T_\infty$, following D. O. Siegmund we adopt the convention that $X_t = \limsup_{n\to\infty} X_n$ if $t = \infty$.

For this class of stochastic sequences D. O. Siegmund has shown (Theorem 4 of [5]), that if:

$$\begin{aligned} s &= \text{first } n \geqq 1 \text{ such that } X_n = f_n \\ &= \infty \quad \text{if no such } n \text{ exists,} \end{aligned}$$

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