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A DELICATE LAW OF THE ITERATED LOGARITHM FOR NON-DECREASING STABLE PROCESSES

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1. Introduction and results. Until quite recently, the only analogues for stable processes of the law of the iterated logarithm were somewhat crude. The difficulty is basically this: from Khinchin's paper [1] it is easy to deduce that for $X(t)$ any stable process of exponent $\alpha < 2$, and $\varphi(t)$ any monotonic function

$$\limsup_{t\to\infty} X(t)/t^{1/\alpha}\varphi(t)$$

is either zero, almost surely, or infinity, almost surely. This was the way the matter rested until Fristed's work in 1964 [2] where he proved, that for $X(t)$ a non-decreasing stable process with $\alpha < 1$,

$$\liminf_{t\to\infty} X(t)/t^{1/\alpha}(\log\log t)^{-\alpha/1-\alpha} = c \quad \text{a.s.}$$

where c is a finite positive constant. This sort of a result we call a delicate law of the iterated logarithm.

Actually, Fristed proved more than the above. For functions $\varphi(t) \downarrow 0$ he almost proved the analogue of the general law of the iterated logarithm by giving conditions on $\varphi(t)$ under which

$$P(X(t) \leqq t^{1/\alpha}\varphi(t) \text{ i.o.} \quad \text{a.s.} \quad t \to \infty)$$

equals zero or one. The two sets of conditions are close together but not the same.

The reason we cannot get a delicate law of the iterated logarithm for lim sup is that the process has upward jumps which are too large. The reason the delicate result holds for lim inf is that whenever $X(t)/t^{1/\alpha}\varphi(t)$ moves downward, it does so continuously. Following Mootoo [3] we can give a simple and elegant proof of the general law of the iterated logarithm which illuminates the above remarks. We make use of a simple time transformation to change $X(t)$ into a recurrent Markov process which has the property that it moves upward only in jumps and downward continuously. Then Mootoo's proof for Brownian motion can be followed, virtually word for word, to give a proof of the following theorem, which is our main result.

THEOREM 1. *Let $X(t)$, $t \geqq 0$, be the non-decreasing stable process of exponent α, $0 < \alpha < 1$. Take $\varphi(t) \downarrow 0$. Then*

$$P(\liminf_{t\to\infty} (X(t) - t^{1/\alpha}\varphi(t)) \leqq 0) = 1$$

if and only if

$$\int_1^\infty [\varphi(t)]^{-\lambda/2} e^{-\mu[\varphi(t)]^{-\lambda}}\, dt/t = \infty$$

where $\lambda = \alpha/(1-\alpha)$ and μ is related to the intensity m of the process, which is de-

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