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EMBEDDING OF URN SCHEMES INTO CONTINUOUS TIME MARKOV BRANCHING PROCESSES AND RELATED LIMIT THEOREMS[1]

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1. Introduction. In this paper we present a technique of embedding certain urn schemes into continuous time Markov branching processes. Typically these urn schemes could be represented by a discrete parameter Markov chain $\{Y_n ; n = 0, 1, 2, \cdots\}$ where the state space is the nonnegative integer lattice in p dimensions for some integer p. We shall establish the existence of certain continuous time Markov branching processes with p-types $\{X(t) = (X_1(t), \cdots, X_p(t)); t \geqq 0\}$ such that for an appropriate sequence τ_n, $n = 0, 1, 2, \cdots$, of increasing stopping times, the stochastic process $\{X(\tau_n); n = 0, 1, 2, \cdots\}$ is equivalent to $\{Y_n, n = 0, 1, 2, \cdots\}$. Thus from limit theorems for $X(t)$ as $t \to \infty$ we can deduce results on the limit behavior of the random variables $\{Y_n\}$ as $n \to \infty$. It turns out that this technique yields many classical and some new results on urn schemes in a relatively simple and more transparent manner.

In this paper we shall be mainly concerned with B. Friedman's scheme (see [7], [8] and Section 2 for a definition). Although we describe the technique in detail only in this case the fundamental idea can easily be adapted to more general situations.

It is interesting to note that urn models have been basic in the study of the spread of contagious diseases and certain ecological and branching processes (see [6], [11]). In this work we proceed in the reverse direction by exploiting properties of branching processes with view to investigate the fluctuation behavior of urn schemes.

An outline of the paper follows. Section 2 introduces the background material concerning multitype continuous time branching processes and reviews some relevant limit theorems for these processes. Section 3 describes the structure of the Friedman and Pólya urn schemes and highlights their connections to branching processes. The principal theorem on the relation of certain results pertaining to multitype continuous time branching processes to those on the embedded Markov chain is contained in Section 4. The applications of this fundamental limit theorem to the case of the Friedman urn are summarized earlier in Section 3.

2. Some results on multitype continuous time Markov branching processes. To make the paper reasonably self contained we devote this section to summarizing results on multitype continuous time Markov branching processes needed later on; for details see ([1], [2], [3]).

Let $\{X(t) = (X_1(t), \cdots, X_p(t)); t \geqq 0\}$ be a p-type continuous time Markov

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