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# AN EXAMPLE OF THE DIFFERENCE BETWEEN THE LÉVY AND LÉVY-PROKHOROV METRICS

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The notation and terminology of [1] will be used below. The purpose of this note is to prove the following:

PROPOSITION. *Let e be a probability measure on* $R^1$ *which is absolutely continuous with respect to Lebesgue measure. Also let* $\mu \rightarrow \mu * e$ *be a 1-1 map of the Borel probability measures into themselves. Let* $\mathfrak{F}$ *be a countable family of Borel probability measures and*

$$\pi = \{\mu * e : \mu \varepsilon \mathfrak{F}\}. \quad \textit{Then:}$$

(i) $\pi$ *is finitely distinguishable if the members of* $\mathfrak{F}$ *are uniformly isolated in the Lévy metric, but the converse does not hold.*

(ii) *If* $\pi$ *is finitely distinguishable then the members of* $\mathfrak{F}$ *are uniformly isolated in the Lévy-Prokhorov metric but the converse does not hold.*

PROOF. Everythring except the converse in (ii) follows immediately from [1]. To construct an example showing the converse does not hold the following combinatorial lemma is needed.

LEMMA. *For any two integers* $0 \leqq m < n$ *there exists a finite set S and subsets* $S_1, \cdots, S_l$ *of S such that (letting* $|B|$ *denote the cardinality of B):*

(1) $|S_i| = n, i = 1, 2, \cdots, l.$

(2) $|S_i \cap S_j| \leqq m$ *if* $i \neq j$.

(3) *If* $C \subseteq S_i$ *and* $|C| = m$, *there exists at least one* $S_j$ *such that* $S_i \cap S_j = C$.

PROOF OF LEMMA. Select an $m + 1$ by $n$ matrix $A$ with entries from the finite field $F_p$ with characteristic $p$ such that any submatrix consisting of $m + 1$ column vectors is nonsingular, i.e., has an inverse matrix over the finite field. This is always possible for given $m$ and $n$ if $p$ is sufficiently large. Let $V$ be the collection of row vectors of length $m + 1$ with entries from $F_p$ and let $Z$ be the corresponding collection of row vectors of length $n$ with entries from $F_p$ which may be written in the form $yA$, $y \varepsilon V$. Since $A$ has rank $m + 1$, $|Z| = |V| = p^{m+1}$. Now let $S = \{(k, q) : k = 1, 2, \cdots, n; q \varepsilon F_p\}$. Each member $z_i$ of $Z$ determines a unique subset $S_i$ of $S$ where $(k, q) \varepsilon S_i$ iff the $k$th coordinate of $z_i$ is $q$. It is clear that $|S_i| = n$. Since every $m + 1$ by $m + 1$ submatrix of $A$ is nonsingular, it follows that for any set of values from $F_p$ for any $m + 1$ coordinates there is one and only one vector $z$ in $Z$ with those coordinate values. From this fact parts (2) and (3) of the lemma follow immediately.

Returning to the construction of an example in (ii) of the proposition we may use the lemma with $n = 2m$, $m = 1, 2, \cdots$, to choose a family $\mathfrak{F}$ of discrete prob-

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