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# NEW CONDITIONS FOR CENTRAL LIMIT THEOREMS[1]

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**1. Introduction.** A general formulation of the central limit problem for sums of independent random variables is the following (see Loève [3], p. 291). Let

$$S_n = \sum_k X_{nk}$$

where $k = 1, \cdots, k_n$, $k_n \to \infty$ as $n \to \infty$, and for each $n$ $\{X_{nk}\}$ are independent random variables with probability distribution functions $F_{nk}$ and $EX_{nk} = 0$. Let $\{F_n\}$ be the distribution functions of $\{S_n\}$ and let $\Phi(x)$ be the distribution function of a normal random variable with zero-mean and variance $\sigma^2$. Under these conditions it is possible to show the following:

THEOREM 1.1. *Let* $\max_k \operatorname{Var} X_{nk} \to 0$ *and* $\sum_k \operatorname{Var} X_{nk} \to \sigma^2 < \infty$ *where* $\sigma^2$ *is a positive constant. The sums* $S_n$ *are asymptotically normal* (i.e., $F_n(x) \to \Phi(x)$) *if and only if for every* $\epsilon > 0$

$$(1.1) \qquad g_n(\epsilon) = \sum_k \int_{|x| \geqq \epsilon} x^2 \, dF_{nk} \to 0.$$

Except in special cases, the application of condition (1.1) is difficult because of the integrals involved. By assuming the existence of fourth-order moments, we are able to prove new necessary and sufficient conditions for both normal and Poisson convergence which involve only moments. The proof of the theorem makes use of a characterization of the normal distribution among infinitely divisible (ID) laws which was perhaps first recognized by Borges [1] and later independently by the author [4].

**2. Normal convergence.**

THEOREM 2.1. *Let* $E|S_n|^{(4+\delta)}$ *be uniformly bounded for some* $\delta > 0$. *Let* $\max_k \operatorname{Var} X_{nk} \to 0$, $\sum_k \operatorname{Var} X_{nk} \to \sigma^2 < \infty$ *where* $\sigma^2$ *is a positive constant. Then* $S_n$ *is asymptotically normal if and only if*

$$(2.1) \qquad ES_n^4 - 3\{ES_n^2\}^2 \to 0.$$

PROOF. The asymptotic normality of $S_n$ implies condition (2.1) by the moment convergence theorem (see Loève [3], p. 184) and the fact that for a zero-mean normal random variable $S$, $ES^4 - 3\{ES^2\}^2 = 0$.

To prove the converse it is sufficient to show that every convergent subsequence $\{F_{n'}\}$ of $\{F_n\}$ converges to $\Phi(x)$ (see Feller [2], p. 261).

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