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EQUIVALENCE OF GAUSSIAN STATIONARY PROCESSES[1]

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Let G be the real line, and $R(s, t) = r(s - t)$ and $S(s, t) = s(s - t)$, where r and s are continuous positive definite functions, and hence by Bochner's theorem can be written as Fourier transforms of finite regular Borel measures (spectral measures) $d\rho$, $d\sigma$. There are unique Gaussian measures P and Q such that the processes defined by $x_t(\omega) = \omega(t)$ have means respectively m and n and covariances respectively R and S. It is known (cf. Grenander [4]) that P and Q are equivalent if and only if

(a) ρ and σ have the identical non-atomic parts;

(b) they have the same set of atoms and if the masses are a_i and b_i then $\sum \{1 - (a_i/b_i)\}^2$ must be finite.

The purpose of this paper is to investigate the necessary and sufficient conditions for P and Q being equivalent when G is a separable locally compact group. Let G be a separable locally compact group which need not be abelian. Then every positive definite function ρ can be written as an integral of positive definite functions ρ_λ

$$\rho(t) = \int_\Lambda \rho_\lambda(t)\, d\mu$$

where Λ may be taken as the real line and μ a finite regular Borel measure. This representation, being essentially unique in the sense that we shall discuss in the paper, is called spectral representation of ρ. Theorem 2 gives the desired conditions in terms of spectral representations.

1. Decomposition of positive definite functions. Let G be a separable locally compact group; $M(G)$, the set of all finite Radon measures on G; γ, a continuous positive definite function on G. Consider the sesqui-linear functional B_γ on $M(G) \times M(G)$ defined by

$$B_\gamma(\alpha, \beta) = \int \int \gamma(s^{-1}t)\alpha(dt)\bar{\beta}(ds) \cdots . \tag{1}$$

It is clear that $B_\gamma(\alpha, \alpha) \geqq 0$ for all $\alpha \varepsilon M(G)$, and the subspace $\mathfrak{N}_\gamma = \{\alpha \varepsilon M(G) | B_\gamma(\alpha, \alpha) = 0\}$ is invariant under the left translations. Let H_γ be the Hilbert space gotten by completing $M(G)/\mathfrak{N}_\gamma$ with the inner product defined by (1). The left translations induce a unitary representation U of G on H_γ in a canonical way.

Let $\mathfrak{U}$ be the von Neumann algebra[2] generated by $\{U_s, s \varepsilon G\}$. Then there is a

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[2] A subalgebra of the Banach algebra of all continuous operators in a Hilbert space is called a von Neumann algebra if it is closed in the weak operator topology, and the ajoint operation. For the general theory in this section we refer to Dixmier [2].