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INVARIANT SETS FOR TRANSLATION-PARAMETER FAMILIES OF MEASURES

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1. Introduction. In this paper we discuss a number of problems which have their origin in statistics but whose main interest is measure-theoretical. It is to the statistician interested in abstract harmonic analysis and to the harmonic analyst interested in statistics that the paper is addressed.

Let $\mathcal{P} = \{P\}$ be a family of probability measures on an arbitrary measurable space $(X, \mathcal{A})$. The set $A \varepsilon \mathcal{A}$ is called '$\mathcal{P}$-invariant' (in preference to the more familiar expression 'similar region') if $P(A)$ is a constant in P. The class $\mathcal{A}(\mathcal{P})$ of $\mathcal{P}$-invariant sets contains all sets that are $\mathcal{P}$-equivalent to the empty set or the whole space, and is closed for complementation and countable disjoint unions. In general, $\mathcal{A}(\mathcal{P})$ is not a sub-σ-field of $\mathcal{A}$.

The set $A \varepsilon \mathcal{A}(\mathcal{P})$ is 'non-trivial' if A is not $\mathcal{P}$-equivalent to the empty set or the whole space. If every member of $\mathcal{A}(\mathcal{P})$ is 'trivial' then we call the family 'weakly complete'. The name is suggested by the fact that 'completeness' $\Rightarrow$ 'bounded completeness' $\Rightarrow$ 'weak completeness'.[3] That weak completeness does not imply bounded completeness is seen from the example where X consists of only three points with a probability distribution θ, θ and $1 - 2\theta$, where $0 < \theta < \frac{1}{2}$.

If $\mathcal{P}$ is not weakly complete, i.e., if there exist non-trivial $\mathcal{P}$-invariant sets, then we call the family of measures 'weakly incomplete.' If $\mathcal{P}$ consists of a finite number of non-atomic measures, then its weak incompleteness is an immediate consequence of a well-known result due to Liapunov [12]. In this situation the class $\mathcal{A}(\mathcal{P})$ is very wide and contains sets of all 'sizes.'

Our main concern is with the weak incompleteness of families of probability measures. Here, we restrict ourselves almost exclusively to the particular situation where $\mathcal{P}$ is a translation parameter family of probability measures. At the risk of some repetitions, this paper brings together a number of results some of which have been noted elsewhere.

2. Notations and a few basic propositions. For the sake of simplicity of exposition we consider the case where X is the additive group of real numbers.

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[3] $\mathcal{P}$ is complete if $\int f\,dP = 0$ for all $P \varepsilon \mathcal{P}$ implies $f = 0$ a.e. $\mathcal{P}$. $\mathcal{P}$ is boundedly complete if f is bounded and $\int f\,dP = 0$ for all $P \varepsilon \mathcal{P}$ implies $f = 0$ a.e. $\mathcal{P}$.