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THE TAIL FIELD OF A MARKOV CHAIN[1]

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1. Introduction. In [1] Blackwell characterized the invariant field of a Markov chain in terms of subsets of the state space called *almost closed sets*. We generalize Blackwell's results, and obtain a similar characterization of the tail field of the chain. Our discussion is modeled upon Chung's exposition ([3], Part I, Sec. 17) of Blackwell's results, and many of our techniques are simple extensions of those to be found in Chung's book.

Let I denote a subset of the integers, which will be the state space of the Markov chain we are going to construct. Let I^∞ denote the space of all sequences $\mathbf{j} = (j_0, j_1, \cdots)$ of elements of I. Let $x_n : I^\infty \to I$ denote the nth coordinate function, $x_n(\mathbf{j}) = j_n$ $(n = 0, 1, \cdots)$. Let $\mathfrak{F}$ denote the smallest Borel field of subsets of I^∞ with respect to which all the functions $x_0, x_1, \cdots$ are measurable.

The shift function $T : I^\infty \to I^\infty$ is defined by setting

$$T(j_0, j_1, \cdots) = (j_1, j_2, \cdots).$$

A set $\Lambda \,\varepsilon\, \mathfrak{F}$ is said to be *invariant* if $T^{-1}\Lambda = \Lambda$. The class of invariant sets, denoted by $\mathcal{G}$, is a σ-field, called the *invariant field*.

If $Y_1, Y_2, \cdots$ is a sequence of functions defined on I^∞, let $\mathcal{B}(Y_1, Y_2, \cdots)$ denote the smallest Borel field with respect to which these functions are measurable. For $n \geqq 0$, let $\mathfrak{F}_n = \mathcal{B}(x_n, x_{n+1}, \cdots)$. We note that $\mathfrak{F} = \mathfrak{F}_0$. Let $\mathfrak{F}_\infty = \bigcap_{n \geqq 0} \mathfrak{F}_n$. $\mathfrak{F}_\infty$ is called the *tail field*. When Λ is a subset of I^∞, by the expression $T\Lambda$ we mean $T\Lambda = \{T\mathbf{j} \mid \mathbf{j} \,\varepsilon\, \Lambda\}$.

THEOREM 1. *T maps $\mathfrak{F}_\infty$ one-to-one onto itself, and* $\mathcal{G} = \{\Lambda \,\varepsilon\, \mathfrak{F}_\infty \mid T\Lambda = \Lambda\}$.

This theorem states that if we regard $\mathfrak{F}_\infty$ as a set of "points," then T acts as a permutation on $\mathfrak{F}_\infty$, and $\mathcal{G}$ is the set of fixed points. Blackwell has shown that modulo equivalence relations, there is an isomorphism between $\mathcal{G}$ and the class of almost closed sets. We will show that the class of almost closed sets can be embedded in a class of objects which is isomorphic in the same way to $\mathfrak{F}_\infty$. Within this class, the almost closed sets correspond to objects which are invariant under the action of a shift function. Furthermore, the isomorphism commutes with this shift.

PROOF OF THEOREM 1. T^{-1} is a countably additive map from $\mathfrak{F}$ into $\mathfrak{F}$, so it is easy to show that $T^{-1}\mathfrak{F}_m = \mathfrak{F}_{m+1}$. For any set $\Lambda \subseteq I^\infty$, $T(T^{-1}\Lambda) = \Lambda$, and so it follows that $T\mathfrak{F}_{m+1} = \mathfrak{F}_m$ $(m \geqq 0)$ and these observations imply that $T\mathfrak{F}_\infty = \mathfrak{F}_\infty$. To show that T is one-to-one, suppose for $\Lambda_1, \Lambda_2 \,\varepsilon\, \mathfrak{F}_\infty$, $T\Lambda_1 = T\Lambda_2$. This means

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