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ON DETECTING CHANGES IN THE MEAN OF NORMAL VARIATES[1]

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A test statistic for the hypothesis of no change in the mean within the first n observations against general alternatives, having prescribed prior distribution, is derived. In the special case of a uniform temporal distribution of at most one change (hereafter abbreviated by AMOC), we prove that the statistic has the same limiting distribution as Smirnov's ω_n^2. Critical values for finite n, obtained by numerical integration, are presented.

1. Introduction and summary. The problem considered is that of detecting changes in the mean of independent unit variance normal random variables when the times of change are assigned an *a priori* distribution. Two situations are considered: The unknown amounts of change are (A) arbitrary, or (B) successively plus and minus the same unknown quantity. Model B is appropriate in certain problems involving angular tracking of an evading target.

In calculating the likelihood of the observations, conditioned on an arbitrary sequence of change indices, we assign the nuisance parameters, viz. the initial mean level and the amount(s) of change, normal probability distributions. Their respective variances are then allowed to approach infinity and zero at appropriate points in the argument. Let $x_1, x_2, \cdots, x_n$ be the first n observations, and let ω_i equal 1 or 0 according to whether there is or is not a change in the mean between x_i and x_{i+1}. In the indicated fashion we find (Sections 2, 3) that the log-likelihood ratios are, up to additive constants,

$$\text{(1.1A)} \qquad \Lambda_{\boldsymbol{\omega}}(x_1, \cdots, x_n) = \sum_{j=1}^{n-1} \omega_j \Big[\sum_{i=j}^{n-1} (x_{i+1} - \bar{x}_n)\Big]^2$$

$$\text{(1.1B)} \qquad = \Big[\sum_{j=1}^{n-1} \Big(\sum_{i=1}^{j} (-1)^{\Omega_i} \omega_i\Big)(x_{j+1} - \bar{x}_n)\Big]^2$$

wherein

$$\Omega_i = 1 + \omega_1 + \cdots + \omega_i,$$

and $\bar{x}_n$ is the arithmetic mean of the first n observations. These ratios are for testing no change against a specified sequence $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_{n-1})$ of change times under Models A and B, respectively. As required, the statistics are translation invariant. A test of the hypothesis of change in the mean at no point, against a set of alternatives $\{\boldsymbol{\omega}\}$ having assigned nonzero prior probabilities, rejects the hypothesis for large values of

$$\text{(1.2)} \qquad Q_n = \sum_{\{\boldsymbol{\omega}\}} p(\boldsymbol{\omega}) \Lambda_{\boldsymbol{\omega}}(x_1, \cdots, x_n).$$

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