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FURTHER REMARKS ON TOPOLOGY AND CONVERGENCE IN SOME ORDERED FAMILIES OF DISTRIBUTIONS

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0. Introduction. In [3] three different types of order relations between distribution functions on the real line were discussed:

(1) $F_1 \leqq F_2$ iff $F_2(t) \leqq F_1(t)$ for all t;
(2) $F_1 \leqq F_2$ iff both
  ($2'$) $F_2(t)/F_1(t)$ is nondecreasing in t, and
  ($2''$) $(1 - F_2(t))/(1 - F_1(t))$ is nondecreasing in t;
(3) $F_1 \leqq F_2$ iff $F_1(t) = 0$ implies $F_2(t) = 0$ and $[F_2(t'') - F_2(t')]/[F_1(t'') - F_1(t')]$ is nondecreasing in both variables t' and t'' whenever $F_1(t') < F_1(t'')$.

It was shown that the order relations (1), (2), (3) are of increasing stringency.

The study of families which are ordered (1) or (2) or (3) is justified by the prominent role which monotone likelihood ratio families play in statistical theory and the fact that monotonicity of likelihood ratios is equivalent to order (3).

Let two distance functions d', d'' be defined by

$$d'(F_1, F_2) = \sup \{|F_1(t) - F_2(t)| : -\infty < t < \infty\},$$
$$d''(F_1, F_2) = \sup \{|P_1(B) - P_2(B)| : B \varepsilon \mathfrak{B}\},$$

where $\mathfrak{B}$ is the Borel-algebra over the real line and P_i the p-measures pertaining to F_i.

In [3] it was shown that $d'(F_1, F_2) \leqq d''(F_1, F_2) \leqq 2(d'(F_1, F_2))^{\frac{1}{2}}$ if $F_1 \leqq F_2$ or $F_2 \leqq F_1$ in the sense of order (2).

Hence the topologies pertaining to these two metrics are equivalent in this case. In view of Lemma (3.1) this immediately implies that the strong and the uniform topology are identical for families which are ordered (2) (Proposition 1.1). It is the purpose of the first part of this paper to extend this result to other topologies. It will be shown that for any family which is ordered (2), the weak and strong neighborhood systems of any nondegenerate distribution are identical (Theorem 1.3). Furthermore it will be shown that for any family of distributions which is ordered (1), the weak topology is equivalent to the induced interval topology. The section concludes with some propositions on connected families.

In the second part of this paper it will be shown that for a dominated family of distributions which is ordered (2), there exists a system of densities such that strong convergence of measures implies convergence of the densities everywhere

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