The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 694–696

NOTE ON SHIFT-INVARIANT SETS

BY U. KRENGEL AND L. SUCHESTON[1]

University of Erlangen and Ohio State University, and The Ohio State University

In this note we prove a theorem which implies that shift-invariant sets in a bilateral product space with infinite invariant measure are contained in the remote σ-algebra (also called tail σ-algebra), if the shift is conservative. After completing the paper we noticed that, in a paper as yet unpublished, K. Dugdale obtained the theorem in the case where the remote σ-algebra is trivial; it seems that his method, based on induced transformations, does not yield our result. W. Parry [6] asserts the same special case of the theorem under some assumptions on the measure space. We further show that in the dissipative case it may happen that the remote σ-algebra is trivial and some invariant sets are not; and that it may also happen that all invariant sets are trivial and the remote σ-algebra is not. Our examples involve transient random walks.

1. Let $(E_k, \mathfrak{F}_k)$, $k = 0, \pm 1, \cdots$, be countably many copies of a measurable space $(E_0, \mathfrak{F}_0)$ and let

$$(\Omega, \mathfrak{A}) = \prod_{k=-\infty}^{+\infty} (E_k, \mathfrak{F}_k).$$

Let X_k be the mapping assigning to the point $\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) \varepsilon \Omega$ its kth coordinate $\omega_k \varepsilon E_k$. The *shift* T on Ω is the transformation defined by $X_k(T\omega) = X_{k+1}(\omega)$. The σ-algebra generated by $X_m, X_{m+1}, \cdots$ is denoted by $\mathfrak{A}_m$. The (right) *remote σ-algebra* $\mathfrak{A}_\infty$ is by definition $\bigcap_{m=0}^{\infty} \mathfrak{A}_m$. A transformation T on Ω is called *invertible* iff T is one-to-one, onto and $A \varepsilon \mathfrak{A}$ implies $T^{-1}A \varepsilon \mathfrak{A}$, $TA \varepsilon \mathfrak{A}$. Clearly, the shift is invertible. Let μ be a fixed measure on $\mathfrak{A}$; henceforth all relations are modulo sets of μ measure zero. An invertible transformation T is called *measure-preserving* iff $A \varepsilon \mathfrak{A}$ implies $\mu(T^{-1}A) = \mu(A) = \mu(TA)$. The σ-algebra of *invariant sets* is defined by: $A \varepsilon \mathfrak{I}$ iff $A \varepsilon \mathfrak{A}$ and $T^{-1}A = A = TA$. A set $A \varepsilon \mathfrak{A}$ is called *wandering* iff the sets $\cdots, T^{-1}A, A, TA, \cdots$ are mutually disjoint. T is called *conservative* iff every wandering set has measure zero.

We state our theorem somewhat abstractly, without reference to the shift. To apply the theorem to sequences $(X_n)_{n=-\infty}^{n=+\infty}$, assume that $\mathfrak{A}_0$ is generated by $X_0, X_1, \cdots$.

THEOREM. *Let T be an invertible conservative measure-preserving transformation on a measure space $(\Omega, \mathfrak{A}, \mu)$ and let $\mathfrak{A}_0$ be a σ-subalgebra such that $T^{-1}\mathfrak{A}_0 \subset \mathfrak{A}_0$ and $\bigcup_{k=0}^{\infty} T^k\mathfrak{A}_0$ generates $\mathfrak{A}$. Assume that μ restricted to $\mathfrak{A}_0$ is σ-finite. Then the σ-algebra $\mathfrak{I}$ of invariant sets is contained in the remote σ-algebra $\mathfrak{A}_\infty = \bigcap_{k=0}^{\infty} T^{-k}\mathfrak{A}_0$, modulo μ-null sets.*

Received 10 April 1968.

[1] Research of this author was supported by the National Science Foundation under grant GP 7693.