The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 692–693

NOTE ON A 'MULTIVARIATE' FORM OF BONFERRONI'S INEQUALITIES[1]

BY R. M. MEYER

State University College of New York at Fredonia

The well-known Bonferroni inequalities (see Feller [1]) provide a sequence of upper and lower bounds on the probability that exactly (or at least) k among n events occur. This note presents an analogous result in the case where one deals with r (finite) classes of events.

For notational convenience derivations are restricted to the case $r = 2$. Let $\{A_1, \cdots, A_M\}$, $\{B_1, \cdots, B_N\}$ be two classes of events. For integers m and n $0 \leqq m \leqq M$, $0 \leqq n \leqq N$ define $P_{[m,n]} = \Pr$ (exactly m A_i's and exactly n B_j's occur). $P_{(m,n)}$ is defined analogously with 'at least' replacing 'exactly.' Let $S_{m,n} = \sum' \Pr(A_{i_1} \cdots A_{i_m} B_{j_1} \cdots B_{j_1})$, where $\sum'$ denotes summation over the indices $1 \leqq i_1 < \cdots < i_m \leqq M$; $1 \leqq j_1 < \cdots < j_n \leqq N$. It is known (see Fréchet [2]) that

$$P_{[m,n]} = \sum_{t=m+n}^{M+N} \sum_{i+j=t} (-1)^{t-(m+n)} \binom{i}{m}\binom{j}{n} S_{i,j} \tag{1}$$

and hence, solving the linear system (1),

$$S_{m,n} = \sum_{t=m+n}^{M+N} \sum_{i+j=t} \binom{i}{m}\binom{j}{n} P_{[i,j]} . \tag{2}$$

A 'bivariate' form of Bonferroni's inequalities is given in the following:

THEOREM 1. *For any non-negative integer* k,

$$\sum_{t=m+n}^{m+n+2k+1} \sum_{i+j=t} f(i, j; t) \leqq P_{[m,n]} \leqq \sum_{t=m+n}^{m+n+2k} \sum_{i+j=t} f(i, j; t), \tag{3}$$

where $f(i, j; t) = (-1)^{t-(m+n)} \binom{i}{m}\binom{j}{n} S_{i,j}$.

PROOF. It suffices to show that $R(r) = \sum_{t=r}^{M+N} \sum_{i+j=t} (-1)^{t-r} \binom{i}{m}\binom{j}{n} S_{i,j} \geqq 0$ for $r \geqq m + n$. Using (2) we have

$$\begin{aligned} R(r) &= \sum_{t=r}^{M+N} \sum_{i+j=t} (-1)^{t-r} \binom{i}{m}\binom{j}{n} \sum_{y=i}^{M} \sum_{z=j}^{N} \binom{y}{i}\binom{z}{j} P_{[y,z]} \\ &= \sum_{i=m}^{M} \sum_{j=r-i}^{N} \sum_{y=i}^{M} \sum_{z=j}^{N} (-1)^{i+j-r} \binom{i}{m}\binom{j}{n}\binom{y}{i}\binom{z}{j} P_{[y,z]} \\ &= \sum_{i=m}^{M} \sum_{y=i}^{M} \sum_{z=r-i}^{N} \binom{z-n-i}{r-i-n-1}\binom{i}{m}\binom{y}{i}\binom{z}{n} P_{[y,z]} \geqq 0. \end{aligned}$$

An analogous result holds for $P_{(m,n)}$. First note that using (1) and an elementary combinatorial identity (for example, 12.8 on page 62 of [1]),

$$P_{(m,n)} = \sum_{y=m}^{M} \sum_{z=n}^{N} P_{[y,z]} = \sum_{i=m}^{M} \sum_{j=n}^{N} (-1)^{i+j-(m+n)} \binom{i-1}{m-1}\binom{j-1}{n-1} S_{i,j} ; \tag{4}$$

hence, by solving the linear system (4) (using 12.13 on page 62 of [1]),

$$S_{i,j} = \sum_{i=m}^{M} \sum_{j=n}^{N} \binom{i-1}{m-1}\binom{j-1}{n-1} P_{(i,j)} . \tag{5}$$

An argument analogous to that used in Theorem 1 yields

Received 27 June 1968.

[1] This work was supported by a National Aeronautics and Space Administration Fellowship at the University of North Carolina at Chapel Hill.