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AN OPTIONAL STOPPING THEOREM

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Let $Y = (Y_n, \mathfrak{F}_n, n \geqq 0)$ denote a stochastic sequence of integrable random variables such that each Y_n is measurable with respect to $\mathfrak{F}_n$, $(\mathfrak{F}_n)$ being an increasing sequence of sub-σ-fields of the σ-field of the underlying probability space. The notion of *stopping time* will be relative to $(\mathfrak{F}_n)$. It may or may not be true that there exists a constant M such that

$$|E[Y_T]| \leqq M \tag{1}$$

for every bounded stopping time T. In Theorem 1 it is shown that for a certain very special class of sequences a suitable M does exist. This fact is used in Theorem 2 to obtain a result in the ergodic theory of Markov chains. Related results have been observed before, see for instance [1], but the present result seems new and the proof is short and intuitive.

Let $X = (X_n, \mathfrak{F}_n, n = 0)$ denote a Markov chain with state space $(S, \mathfrak{B})$ and stationary transition probability function $P(x, B)$. Write $P^k(x, A)$ for the k-step transition probability function.

THEOREM 1. *Let g be a real valued measurable function on $(S, \mathfrak{B})$ which is bounded in absolute value. Suppose*

$$Gg(x) = \lim_{n\to\infty} \textstyle\sum_{k=1}^n (P^k g)(x)$$

exists for all $x \varepsilon S$, and that Gg is bounded in absolute value. Let $Y = (Y_n, \mathfrak{F}_n, n \geqq 0)$, $Y_n = \sum_{k=0}^n g(X_k)$. Then there exists an M such that (1) *holds for every bounded stopping time T.*

PROOF. Write

$$Z_n = \textstyle\sum_{k=0}^n g(X_k) + Gg(X_n)$$

and observe that $Z = (Z_n, \mathfrak{F}_n, n \geqq 0)$ is a martingale. For bounded stopping times T the optional sampling theorem for martingales implies $E[Z_T] = E[Z_0]$, hence

$$|E[Y_T] - E[Z_0]| \leqq \sup_x |Gg(x)|.$$

The Markov chain X is *recurrent in the sense of Harris* if there exists a σ-finite measure π on $(S, \mathfrak{B})$ such that $\pi = \pi P$ and $B \varepsilon \mathfrak{B}$, $\pi(B) > 0$ implies

$$\lim_{n\to\infty} P_x[\textstyle\bigcup_{k=0}^n [X_k \varepsilon B]] = 1, \qquad x \varepsilon S. \tag{2}$$

If the convergence in (2) is uniform in x, X is *uniformly recurrent*. The measure π is unique up to a constant factor, and in the uniformly recurrent case it is necessarily finite and will be assumed to be normalized to be a probability measure.

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